Graph Formation Effects on Social Welfare and Inequality in a Networked Resource Game

Zhuoshu Li[1], Yu-Han Chang[2], and Rajiv Maheswaran[2]

[1]Beihang University, Beijing, China
[2]Information Sciences Institute, University of Southern California, CA, USA
zslibuaa@gmail.com, ychang@isi.edu, maheswar@isi.edu

Abstract. We introduce the Networked Resource Game, a graphical game where players' actions are a set of resources that they can apply over links in a graph to form partnerships that yield rewards. This introduces a new constraint on actions over multiple links. We investigate several network formation algorithms and find bilateral coalition-proof equilibria for these games. We analyze the outcomes in terms of social welfare and inequality, as measured by the Gini coefficient, and show how graph formation affects these aspects of a networked economy.

1 Introduction

Graphical games that model social phenomena have been an emerging research area applied to group consensus making, networked bargaining and trading strategies. Here, we investigate the interactions of a society where actions are resource-bounded, i.e., agents have limits on how they are able to act across their network. We model the notion that people have a finite number of resources and their network affects how those resources can be coupled with others' resources in order to produce rewards. One example of this is in professional networks where agents need to form partnerships and the payoffs of the partnerships are a function of the capabilities that each bring to the table.

In this paper, we introduce the *Networked Resource Game*, and study how the structure of the network and its dynamics affect social welfare and inequality, measured by the Gini coefficient, of the resulting equilibria. For network formation, we utilize Erdos-Renyi [13] and preferential attachment [1] models and introduce several new algorithms as well. We introduce an algorithm to find bilateral coalition-proof equilibria as Nash equilibria do not lead to reasonable outcomes in this domain. In this context, we study how the various algorithms affect social welfare and inequality and the impact of network properties on performance.

2 Related Work

Graphical games [9] provide compact representation of multi-agent interaction when players' payoffs depend only on actions of agents in their neighborhood.

It is known that finding Nash equilibria for graphical games is difficult even for restricted structures [4]. Local heuristic techniques are commonly employed [7, 3]. A seminal work in using agent-based simulation to study human interaction was Axelrod's tournament for Prisoner's Dilemma [2]. Prisoner's Dilemma has also been studied in a graphical setting with simulated agents [11]. Dynamic networked games based on the Ultimatum Game have also been investigated [10] Research on identification and development of networks includes analyzing event-driven growth [14] and inferring social situations by interaction geometry [6]. Other work has described algorithmic methods to discover temporal patterns in networked interaction data [8]. Researchers have formulated efficient solution methods for games with special structures, such as limited degree of interactions between players linked in a network, or limited influence of their action choices on overall payoffs for all players [12, 15]. In terms of these work, our model takes the networked interaction into a completely different domain, as we focus on the influence of the structure and topology of the network, on the dynamics of resource allocation in the network.

3 Networked Resource Game Model

The Networked Resource Game is characterized by a set of N players $\{p_i\}_{i=1}^N$, a card distribution C, a graph G and a reward function R. Each player p_i has a set of cards $C_i = \{c_{i,1}, \ldots, c_{i,N_i^C}\}$ where N_i^C is the number of cards for that player. The cards represent a skill or resource that the player can play on a link. Each card has a type which comes from a predetermined type set T, i.e., $c_{i,j} \in T \;\; \forall i, j$. For simplicity, given a discrete type set, we can think of the type as a color and that each card has a color. The graph is a set of edges over N nodes, i.e., $G = \{e_{ij}\}$ where e_{ij} refers to a *link* between players p_i and p_j. It is possible that some players have no links associated with them.

The graph specifies the links over which players may play their cards. Here, we include the restriction that a player may play at most one card on a link. Thus,

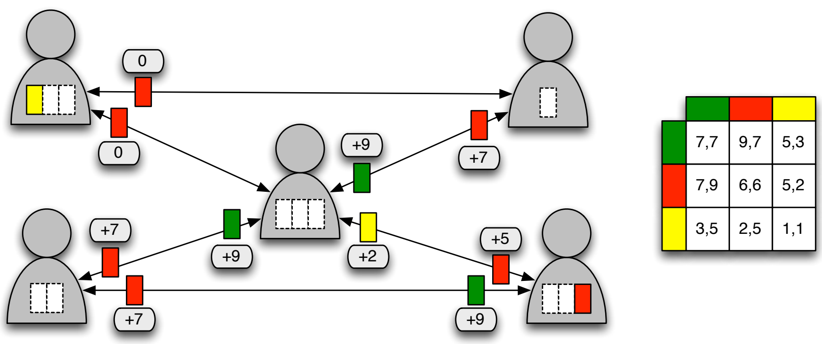

Fig. 1. The Networked Resource Game

the number of cards indicate a players ability to have multiple simultaneous partnerships. It is possible that a player has more links than cards and also more cards than links. Based on what cards are played on a link, each player gets a reward specified by the function $R(a, \bar{a})$ which is the reward to a player for performing action a on a link where the other player performed action $\bar{a}$. The action space for player p_i on link e_{ij} is $A_{ij} = C_i \cup 0$, where 0 indicates that the player chose not to play one of their cards on that link.

The reward function has $(|T| + 1)^2$ inputs representing every combination of actions, i.e., all card types and not playing a card, for each player. The Networked Resource Game is similar to a standard graphical game, however, the action space has restrictions over multiple links whereas in standard graphical games, actions on link are independent. Here, we have the restriction that $\cup_j a_{ij} \subset C_i$ where a_{ij} is player p_i's action on link e_{ij}. This states that a player cannot play more cards than they have, which introduces a coupling over links.

An illustration of the game is shown in Figure 1. It shows a game involving three card types (green, red and yellow). One can imagine that these cards represent assets of value in an economy that yield different outcomes to each contributor in partnerships. For example, green could represent capital, red could represent skilled labor and yellow could represent unskilled labor.

4 Network Formation and Finding Equilibria

Network Formation. Here, we describe the algorithms that we use to create our social network graphs and find equilibria for a given graph. Network formation is determined by various growth processes that describe how a link is added to an existing graph. We describe four such models:

- **Erdos-Renyi (ER)**: This is a baseline process where we add a link chosen uniformly from those links that do not already exist in the graph.
- **Preferential Attachment (PA)**: If the input graph has zero or one link, we use the ER process. Thus, the network is seeded with two random links. After this, to add a link, we choose a node randomly and consider the links it could add to the graph, i.e., the set of links connected to the chosen node that are not already in the graph. Each such link is given a weight equal to the degree of the target node it connects to, and a link is chosen in proportion to these weights. Preferential attachment models have been proposed as a model that reflects how social networks are formed, particularly online.
- **Most Free Cards (MFC)**: Each node is given an MFC score: the number of cards it has minus the number of links it has, i.e., a measure of the number of free cards for that player. The process selects a node uniformly from those that have the highest MFC score. This node then chooses a link uniformly from other nodes that have the highest MFC score. When the MFC scores are all zero, the algorithm becomes ER.
- **Poor-to-Rich Chain (PRC)**: We first associate each player with a wealth calculated as the sum of the value of their cards, where the value of each

card is the maximum reward obtainable from applying that card:

$$w_i = \sum_{c \in C_i} \max_{a \in T \cup 0} R(c, a)$$

We first create a chain, where agents are ordered by wealth with ties broken randomly. Then, a player chosen uniformly from those with the highest MFC score adds a link. The target node is the closest node in the chain with a free card, i.e., an MFC score greater than zero. Again, ties are broken randomly. When all MFC scores are zero, the algorithm becomes ER.

The various processes described above capture various degrees of control that players may have over the network on which they play. In the ER and PA models, players have no control over links. One may consider PA as player driven, but the game properties (card, rewards) do not affect the formation of the links so the processes are not strategic. The MFC model is a decentralized strategic model where agents have partial information about the state of the world, namely the number of cards and links for each player. The PRC model is a centralized model that takes game parameters into account when making the graph and incorporates a social structure onto the world where people with similar wealth are more likely to be connected to each other.

Finding Equilibria. Given a game structure (cards, rewards, and a graph), we would like to determine an appropriate outcome. Nash equilibria are often considered as a solution concept for games and graphical games as well, however, it has issues in the context of the Networked Resource Game. Consider the simple example of four players in a fully connected graph where each player has one card. Two players have a single red card and two players have a single green card. Let the rewards for having two cards with same color on a link be 100 for each player, two cards with different colors on the same link be 10, and all links with one or zero links be worth nothing. Consider the situation where we have two red-green links. Each player receives 10 and has no incentive to deviate, i.e, move their one card to another link, because that would cause a loss of 10, even though each player has a link to a player with the same color card.

Thus, in the Networked Resource Game, Nash equilibria lead to artificially poorer results than one would expect if one was playing this game assuming players could communicate over the links that they have. Thus, we consider equilibria where players can make bilateral deviations. An equilibrium in this context is a state where no player would choose to make a unilateral deviation and no two players would choose to make a bilateral deviation. We use the procedure below to discover such equilibria for a given game structure.

Each player first assigns cards randomly to available links. We then perform action updates in a series of rounds. In each round, we order the set of links. For each link, the players iterate back and forth on card choices for the link. On the first iteration, the first player assumes that the other player plays one of their cards, chosen from all cards that player has, i.e., not necessarily the card being played on the link currently. The first player then plays their best response on all links given the cards that are played on all the links that they have. In the second iteration and all following iterations, the acting player chooses their best response to the cards that are being played on their link. This procedure continues until an equilibrium is reached for that link or we reach a preset limit of interactions. We continue this procedure for all links in each round. The procedure terminates, when at the end of a round, the joint actions are the same as the joint actions in the previous round. The procedure continues for a preset number of rounds. Finding equilibria in graphical games is a challenging problem. The algorithm presented is sound in that if it terminates before reaching the preset number of rounds, we know that the resulting joint action is an equilibrium for the game, however, we may not find all equilibria.

Abstract algorithm FINDING-EQUILIBRIA for computing bilateral coalition-proof equilibria

```
Algorithm FINDING-EQUILIBRIA
Inputs: one game structure(cards,rewards and a graph)
Outputs: bilateral coalition-proof equilibria
  Each player first assigns cards randomly to available links
  equilibria ← ∅
  for round ← 1 to n1
    do order the set of links
    num ← 1
    repeat
      (1)one player P_i assumes that the other player P_j
         which links with P_i plays one of its cards
      (2)P_i plays their best response on all links given the
         cards that are played on all the links that they have
      (3)P_j chooses their best response to the cards
         that are being played on their link
      (4)num ← num + 1
    until an equilibrium is reached for that link
          or num = n2
    if the joint actions = joint actions in the previous round
    then return equilibria
    else return -1
```

5 Experiments

We considered societies of 12 players. In each scenario, each player was given a number of cards chosen uniformly from one to five: $|C_i| \sim U(1,5)$. We had three card types: green, red, and yellow. Card colors were selected independently for each card using the following probabilities: $P([\text{green red yellow}]) = [0.20\ 0.40\ 0.40]$. There were two methods for selecting reward functions. In the baseline method, each reward for links with two cards on them were chosen randomly: $R(c_1, c_2) \sim U(1, 1000)$ for $c_1, c_2 \in T$. Links with one or zero cards gave zero reward to both players. In the alternate method, the reward for a green-green link is replaced with 100 times the value of the maximum reward of all the rewards in the baseline method. The latter is to investigate a society where there is a significantly outlying reward available to a small number of people if they make the right connections. It is for this reason that the green cards occur at lower likelihood than the others. For a given game card and reward structure, we would run our various network formation algorithms and generate graphs of increasing size. Each network formation algorithm was run 10 times, thus generating 10 graphs with the same number of edges for each process. For each game structure (cards, rewards and graph) that resulted, we would find the set of equilibria. For each graph, the equilibrium-finding algorithm was run 40 times and each run was ended if the algorithm didn't terminate in 15 rounds.

For any single equilibrium, we calculated the *social welfare* as the sum of all the rewards to all players and the *Gini coefficient*, a measure of income disparity [5, 16]. The Gini coefficient measures the gap in the cumulative distribution function (CDF) of total share of wealth as a function of percentile income between a uniformly wealthy society which would have a linear CDF and the CDF of the society being investigated. Larger Gini coefficients indicate greater income disparity. For each game structure, we calculated an associated social welfare with the weighted average of social welfares of equilibria of that game structure, where weights were the number of times the equilibrium was discovered. We calculated associated Gini coefficients for each game structure similarly.

The Gini coefficient is normalized between zero (everyone has equal wealth) and one (one person has all the wealth), but social welfare for each game is a function of the reward matrix. We first solve the following integer program:

$$\begin{aligned} \max \sum_{(c_1,c_2)\in C_2} & n_{c_1,c_2}\left(R(c_1,c_2)+R(c_2,c_1)\right) \\ \text{such that } \sum_{\tilde{c}\in T} & n_{c,\tilde{c}} \leq n_c \quad \forall c\in T, \quad n_{c,\tilde{c}} \geq 0, \quad \forall c,\tilde{c} \end{aligned}$$

This considers all possible combinations of cards on a link $(c_1, c_2) \in C_2$ and maximizes the reward obtained for having a particular number of card combinations on the graph (n_{c_1,c_2}) with the rewards obtained for that card combination $(R(c_1, c_2) + R(c_2, c_1))$, such that the number of card combinations of the graph does not violate the card constraints, i.e., the number of cards of a particular type (n_c) and non-negativity of the number of combinations. This yields an upper bound on the social welfare because it allows multiple links between players and links between cards of the same player. We use this to normalize social welfares across different card and reward structures.

6 Results

Figure 2 shows how social welfare changes as a function of network formation algorithm and graph size. We did not show the error bars for clarity in presentation but we discuss significance below. We see that social welfare improves as the society gets more connected for all algorithms. MFC and PRC are significantly better than ER and PA. ER is slightly better than PA but the result is not statistically significant. These results hold in both reward scenarios. For baseline rewards, MFC and PRC both reach about 0.9 efficiency in social welfare at about 18 links and do not improve much beyond that. We also see the impact of network structure as the 28-link ER and PA graphs are less efficient that MFC and PRC graphs that are half the size. We noticed that ER and PA graphs are not easy to reach equilibria when the graph is larger than 30 links. For alternate rewards, the efficiency is significantly smaller than the baseline word, this could be the result of two factors: there are green-green links that are not being formed, and our normalization could be overcounting the number of potential green-green links.

Figure 3(a) shows how Gini coefficients change as a function of network formation algorithm and graph size. Inequality decreases as the network sizes increase. For the baseline reward structure, MFC, PRC and ER are significantly better than PA. The key change is that ER has jumped from the PA equivalence class to the MFC/PRC equivalence class. We note that the Gini coefficient is relatively flat after about 18 links. For the alternate reward structure, all the algorithms are in the same equivalence class. This is because once a few green-green links are formed, it is difficult to change the inequality of the world.

We then investigated the number of wasted cards in equilibrium, i.e., the number of cards that did not yield any reward to the player holding it. Figure 3(b) shows the number of wasted cards as a percentage of the total number of cards in a society. We see that wasted cards explains a lot of the phenomena in social welfare. The MFC and PRC algorithm, which has an MFC component, waste the fewest cards because that is part of their process. The others form links that are not as useful in allowing players to use their cards. ER performs slightly better that PA because it does not overload particular users with large numbers of links. Thus as fewer cards are wasted, social welfare improves. This similarly explains the Gini coefficient because as more cards are used, we have fewer users with low or no rewards. Nevertheless, it is interesting to note that

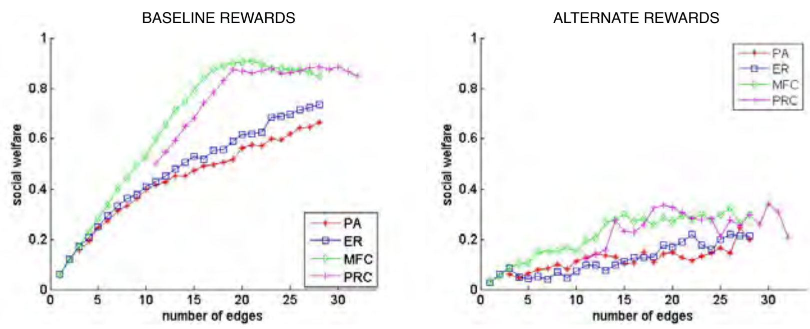

Fig. 2. Social Welfare by Algorithm and Graph Size

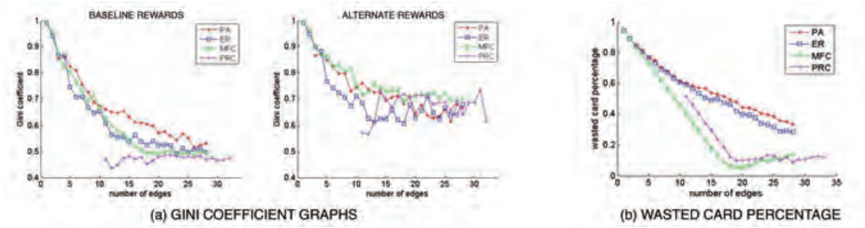

Fig. 3. (a) Gini Coefficient and (b) Wasted Card Percentage by Algorithm, Graph Size

while ER wastes more cards than MFC and PRC, it does not perform worse in terms of inequality. This remains an open question. Interestingly, with half the possible links (33), we still have about 10% of cards being wasted.

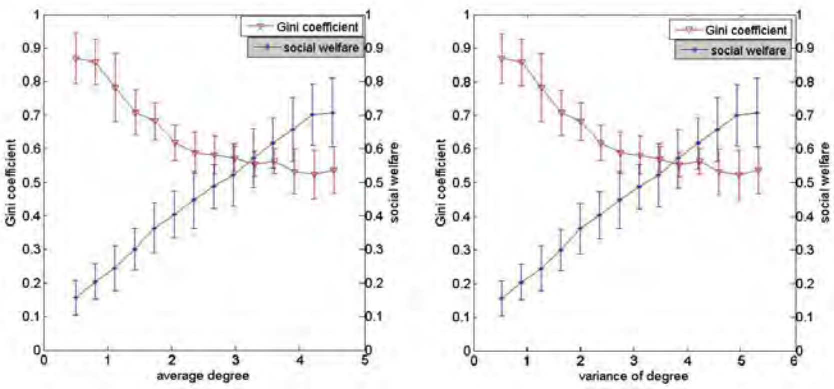

Fig. 4. Social Welfare and Gini Coefficient by Average and Variance of Degree for Baseline Rewards

We also looked at the impact of network properties on outcomes. Figure 5 shows social welfare and Gini coefficient as a function of the average and variance of the degrees of the nodes in the graph. Clearly, this will depend on the card and reward structure. In our case, both average and variance of degree showed similar curves in increasing social welfare and decreasing inequality. The inequality curves are similar in both reward structures and the social welfare curves are close to the best performing algorithms as a function of graph size. One potential future direction is using these properties as part of the network formation process because they may be more easily estimated than the requirements of the processes we presented. We also plan on investigating games where more than two players can collaborate. It is also a challenge to investigate appropriate outcomes for graphs as the scale of the society grows as equilibrium discovery will become more computationally demanding. We believe the Networked Resource

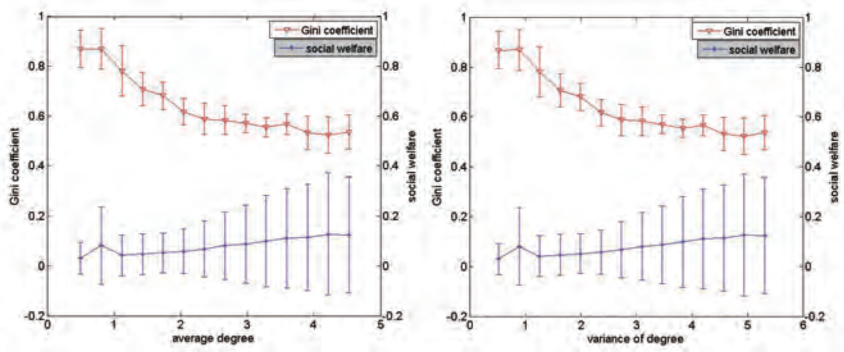

Fig. 5. Social Welfare and Gini Coefficient by Average and Variance of Degree for Alternate Rewards

Game is a good starting point for modeling and investigating the complexities and design of economies of resource-bounded and socially networked agents.

Acknowledgements

This material is based upon work supported in part by the AFOSR under Award No. FA9550-10-1-0569.

References

1. R. Albert and A.-L. Barabasi. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 74:47–97, 01 2002.
2. R. Axelrod and W. D. Hamilton. The evolution of cooperation. *Science*, 211:1390–96, 1981.
3. Q. Duong, Y. Vorobeychik, S. Singh, and M. P. Wellman. Learning graphical game models. In *IJCAI*, 2011.
4. E. Elkind, L. Goldberg, and P. Goldberg. Nash equilibria in graphical games on trees revisited. In *Proceedings of the 7th ACM conference on Electronic commerce*, pages 100–109. ACM, 2006.
5. C. Gini. On the measure of concentration with special reference to income and statistics. *Colorado College Publication*, 1936.
6. G. Groh, A. Lehmann, J. Reimers, M. Friess, and L. Schwarz. Detecting social situations from interaction geometry. In *Social Computing (SocialCom), 2010 IEEE Second International Conference on*, pages 1 –8, Aug. 2010.
7. D. Heckerman, D. Geiger, and D. M. Chickering. Learning bayesian networks: The combination of knowledge and statistical data. In *Machine Learning*, pages 197–243, 1995.
8. H.-P. Hsieh and C.-T. Li. Mining temporal subgraph patterns in heterogeneous information networks. In *Social Computing (SocialCom), 2010 IEEE Second International Conference on*, pages 282 –287, Aug. 2010.

9. M. Kearns, M. Littman, and S. Singh. Graphical models for game theory. In *Conference on Uncertainty in Artificial Intelligence*, pages 253–260, 2001.
10. E. Kim, L. Chi, R. Maheswaran, and Y.-H. Chang. Dynamics of behavior in a network game. In *IEEE International Conference on Social Computation*, 2011.
11. L. Luo, N. Chakraborty, and K. Sycara. Prisoner's dilemma in graphs with heterogeneous agents. In *Social Computing (SocialCom), 2010 IEEE Second International Conference on*, pages 145 –152, Aug. 2010.
12. L. Ortiz and M. Kearns. Nash propagation for loopy graphical games. In *Neural Information Processing Systems*, 2003.
13. E. P. and R. A. On random graphs. i. *Publicationes Mathematicae*, 6:290–297, 1959.
14. B. Qiu, K. Ivanova, J. Yen, and P. Liu. Behavior evolution and event-driven growth dynamics in social networks. In *Social Computing (SocialCom), 2010 IEEE Second International Conference on*, pages 217 –224, Aug. 2010.
15. D. Vickrey and D. Koller. Multi-agent algorithms for solving graphical games. In *National Conference on Artificial Intelligence (AAAI)*, 2002.
16. S. Yitzhaki. More than a dozen alternative ways of spelling gini economic inequality. *Economic Inequality*, 1998.