A Model of Economic Mobility and the Distribution of Wealth

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Abstract

This paper introduces new techniques to obtain a closed-form rank-by-rank characterization of the equilibrium distribution of wealth in a model in which finitely lived households face uninsurable idiosyncratic investment risk. A central result is that the extent of inequality is determined entirely by two factors. The first factor, household exposure to idiosyncratic investment risk, increases inequality. The second factor, cross-sectional mean reversion of household wealth, decreases inequality. We show that economic mobility is decreasing in inequality and increasing in mean reversion, a result that is consistent with recent empirical observations about the geographic variation in mobility that exists both domestically and internationally. Our approach allows us to examine the implications of increased market completeness in the form of a risk-sharing subgroup of households. We show that a risk-sharing subgroup rises or falls in the equilibrium wealth distribution depending on the level of inequality, and that its presence raises welfare and the rate of wealth accumulation for all households in the economy.

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1 Introduction

A large collection of recent research has described the substantial concentration of income and wealth that exists in many different countries around the world. According to Davies et al. (2011) and Saez and Zucman (2014), the wealthiest 1% of households in the United States hold approximately 40% of total wealth, with similar numbers observed in other countries as well. Atkinson et al. (2011) show that the wealthiest 1% of households in the United States earn nearly 25% of total income, and that many other countries have similarly right-skewed distributions of income.[1] Although right-skewed distributions of income and wealth have been the reality for generations, the extent of such skewness has varied markedly over time and across countries.

In order to explore how the economic environment and policy influence mobility and the distributions of income and wealth, we develop a model in which finitely lived heterogeneous households face uninsurable idiosyncratic investment risk and have a joy-of-giving bequest motive. Using a novel rank-based approach to wealth distribution, we are able to obtain a closed-form rank-by-rank characterization of the entire equilibrium distribution of wealth in this setting. According to this characterization, the extent of economic inequality is determined entirely by two factors—the reversion rates of household wealth (a measure of cross-sectional mean reversion) and households' exposure to idiosyncratic investment risk. Mean reversion in our model is determined by households' finite lifetimes and the lump sum government transfers paid to all newborn households. In the steady-state equilibrium, inequality is increasing in household exposure to investment risk and decreasing in the reversion rates.

In addition to inequality, our solution techniques also allow for a detailed analysis of economic mobility. We show analytically that mobility is decreasing in inequality and increasing in the reversion rates, a result that is broadly consistent with both the "Great Gatsby curve" documented across countries (Krueger, 2012; Corak, 2013) and the geographical variation in mobility documented within the U.S. (Chetty et al., 2014). We also consider the implications of increased market completeness in the form of a subgroup of households that can share their individual-specific idiosyncratic investment risk with each other. Even in the absence

[1] Díaz-Giménez et al. (2002) provide further analysis of the distribution of income in the United States. Meanwhile, Piketty (2003), Piketty and Saez (2003), Atkinson (2005), Saez and Veall (2005), and Moriguchi and Saez (2008) present detailed analyses of income inequality in France, the United States, the United Kingdom, Canada, and Japan, respectively.

of changes in investment and consumption in response to their increased investment opportunities, the households in a risk-sharing subgroup achieve both faster wealth accumulation and less volatile returns. In equilibrium, this faster wealth accumulation spreads to those households outside the risk-sharing subgroup via larger lump sum transfers, thus raising the welfare of all households in the economy.

Many of the first attempts to account for the right-skewness of the distribution of wealth assumed that households face uninsurable idiosyncratic labor income risk. While this approach has had some empirical success, many of these so-called Bewley models fail to generate high Gini coefficients and heavily right-skewed wealth distributions.[2] Another explanation for right-skewed distributions of wealth involves uninsurable investment risk and the multiplicative process of wealth accumulation. The assumption that households face idiosyncratic investment risk was first introduced into a macroeconomic model by Angeletos and Calvet (2006) and Angeletos (2007), and has since been incorporated into models of wealth distribution such as Benhabib and Zhu (2009) and Benhabib et al. (2011).

The primary motivation for the inclusion of uninsurable investment risk is empirical. Indeed, ample evidence demonstrates that both private business equity and principal residence ownership are important sources of idiosyncratic investment risk for individuals and households. According to calculations by Bertaut and Starr-McCluer (2002) and Wolff (2006) using data from the 2001 *Survey of Consumer Finances* (SCF), private business equity and the gross value of principal residences make up, respectively, 27% and 28.2% of total U.S. household wealth. These investments are highly volatile, with a standard deviation for the return of housing roughly equal to 15% according to Case and Shiller (1989) and Flavin and Yamashita (2002), and an even larger volatility for the capital gains and earnings on private equity as reported by Moskowitz and Vissing-Jorgensen (2002). Taken together, these facts imply that a substantial fraction of risk associated with the returns to household wealth is not easily diversified away.

This paper joins a growing literature that embraces these empirical insights and incorporates uninsurable investment risk in a macroeconomic setting. In particular, our setup shares much in common with the subset of this literature focused on wealth distribution,

[2] Ljungqvist and Sargent (2004) and Cagetti and De Nardi (2008) provide both a discussion and survey of this extensive literature. Some authors have in fact successfully generated right-skewness in this setting by expanding the setup to include extra features such as borrowing constraints, preferences for bequests, and entrepreneurship (Cagetti and De Nardi (2006), De Nardi (2004), and Quadrini (2000)) or heterogeneous and fluctuating discount rates (Krussel and Smith (1998) and Hendricks (2007)).

such as Benhabib et al. (2011, 2014). Like these papers, our model generates an equilibrium distribution of wealth that is right-skewed and has a Pareto-like shape that matches what is observed in real-world economies.

In addition to describing the distribution of wealth, the results in this paper extend the literature in three directions. First, we adopt the general, nonparametric approach to rank-based systems introduced by Fernholz (2016) and use this to achieve a simple analytic characterization of equilibrium in our model. In most cases, the technical challenges posed by uninsurable investment risk and the multiplicative process of wealth accumulation essentially preclude a closed-form characterization of the distribution of wealth. Even in those cases where an analytic solution is possible, the complexity of such a solution often makes it difficult to interpret or understand. The equilibrium characterization in our setup, by contrast, is both tractable and flexible. In it, there are just two factors—households' exposure to idiosyncratic investment risk and the reversion rates of household wealth—that determine the entire distribution of wealth and inequality according to the relationship[3]

$$\text{inequality} = \frac{\text{household exposure to idiosyncratic investment risk}}{\text{reversion rates of household wealth}}. \tag{1.1}$$

As this equation shows, equilibrium inequality is increasing in exposure to investment risk and decreasing in cross-sectional mean reversion (measured by the reversion rates).

This paper's second contribution is a detailed description of the extent of mobility in the economy. In our setup, we model explicitly heterogeneous households whose wealth dynamics over time are traced individually. One natural measure of economic mobility in this setting is the expected time before one household overtakes another household in terms of wealth holdings. After all, if one household overtakes another in this way, then that household has improved its economic standing, which is the usual measure of mobility. A central result in this paper characterizes the expected time before each such improvement in household economic standing occurs. In particular, we show that this measure of mobility is entirely determined by two factors according to the relationship

$$\text{mobility} = \frac{\text{reversion rates of household wealth}}{\text{inequality}}. \tag{1.2}$$

[3]The reversion rates in our setup are closely related to the concept of redistributive mechanisms from Fernholz and Fernholz (2014).

Much like inequality, then, equilibrium mobility is decreasing in cross-sectional mean reversion and increasing in inequality.

In addition to being supported by the empirical results of Corak (2013) and Chetty et al. (2014), these theoretical predictions provide an interpretation of the variation in mobility that has been observed across different regions, countries, and time periods. For instance, according to our results, the substantial variation in intergenerational income mobility across different geographical areas of the United States documented by Chetty et al. (2014) should reflect a similar geographic variation in inequality and reversion rates of household wealth. A similar conclusion applies to the variation in mobility observed across countries by Corak (2013). Ultimately, it is likely that there are many different factors, both related and unrelated to policy, that explain such variation in inequality and mean reversion across regions and countries. Future empirical research that attempts to distinguish among these factors should yield useful insight.

Our third contribution is to examine the implications for the equilibrium distribution of wealth and welfare of increased market completeness. In particular, we consider the presence of a subgroup of households that can share their individual-specific idiosyncratic risk with each other and show that this has two separate effects. First, the households in the subgroup are able to achieve less volatile returns and faster wealth accumulation simply by diversifying across different risky assets. Second, because the households in the subgroup have better risky-asset investment options, they invest more aggressively in higher-return risky assets and thus achieve even faster wealth accumulation. This faster wealth accumulation spreads to those households outside the risk-sharing subgroup as well since it increases the size of lump sum transfers to newborn households in equilibrium. As a consequence, the presence of a risk-sharing subgroup of households leads to faster wealth accumulation and higher welfare for all households in the economy. In terms of the equilibrium distribution of wealth, we show that the more inequality there is, the higher the risk-sharing subgroup of households rises in the distribution.

The rest of the paper is organized as follows. Section 2 introduces the general, rank-based solution techniques that we use to obtain our results. Section 3 presents the main model and the results about the distribution of wealth, economic mobility, and the implications of risk-sharing subgroups of households. In Section 4, we parameterize the model to match the most recent U.S. wealth distribution data from 2012 and generate numerical results for intergenerational mobility, rank correlations, and the dynamic behavior of a risk-sharing subgroup. Section 5 concludes. All proofs can be found in Appendix A.

2 Preliminaries: A Nonparametric Approach to Wealth Distribution

Before we present and solve our theoretical model of wealth distribution, it is useful to present some basic preliminary results that motivate our approach to solving the theoretical model.

Consider an economy that is populated by $N > 1$ households. Time is continuous and denoted by $t \in [0, \infty)$, and uncertainty in this economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let $\mathbf{B}(t) = (B_1(t), \ldots, B_M(t))$, $t \in [0, \infty)$, be an M-dimensional Brownian motion defined on the probability space, with $M \geq N$. We assume that all stochastic processes are adapted to $\{\mathcal{F}_t; t \in [0, \infty)\}$, the augmented filtration generated by $\mathbf{B}$.

The total wealth of each household $i = 1, \ldots, N$ in this economy is given by the process w_i. Each of these wealth processes evolves according to the stochastic differential equation

$$d \log w_i(t) = \mu_i(t)\, dt + \sum_{z=1}^{M} \delta_{iz}(t)\, dB_z(t), \tag{2.1}$$

where μ_i and δ_{iz}, $z = 1, \ldots, M$, are measurable and adapted processes. The growth rates and volatilities μ_i and δ_{iz} are general and can vary over time and across different household characteristics.[4] Indeed, equation (2.1) requires only that the wealth processes for the banks in the economy be continuous semimartingales, which represent a broad class of stochastic processes (for a detailed discussion, see Karatzas and Shreve, 1991). Indeed, the martingale representation theorem (Nielsen, 1999) implies that any plausible continuous wealth process can be written in the nonparametric form of equation 2.1.[5]

[4] These growth rates and volatilities must only satisfy a few basic regularity conditions that are discussed in detail by Fernholz (2016).

[5] This is even true of wealth processes that feature periodic discontinuous jumps, as we demonstrate in the next section.

It is useful to describe the dynamics of total wealth for all households in the economy, which we denote by $w(t) = w_1(t) + \cdots + w_N(t)$. In order to do so, we first characterize the covariance of wealth across different households over time. For all $i, j = 1, \ldots, N$, let the covariance process ρ_{ij} be given by

$$\rho_{ij}(t) = \sum_{z=1}^{M} \delta_{iz}(t)\delta_{jz}(t). \tag{2.2}$$

Applying Itô's Lemma to equation (2.1), Fernholz (2016) shows that the dynamics of the process for total wealth in the economy w are given by

$$d\log w(t) = \mu(t)\,dt + \sum_{i=1}^{N}\sum_{z=1}^{M} \theta_i(t)\delta_{iz}(t)\,dB_z(t), \quad \text{a.s.}, \tag{2.3}$$

where

$$\mu(t) = \sum_{i=1}^{N} \theta_i(t)\mu_i(t) + \frac{1}{2}\left(\sum_{i=1}^{N} \theta_i(t)\rho_{ii}(t) - \sum_{i,j=1}^{N} \theta_i(t)\theta_j(t)\rho_{ij}(t)\right) \tag{2.4}$$

and, for all $i = 1, \ldots, N$, $\theta_i(t)$ is the share of total wealth held by household i at time t,

$$\theta_i(t) = \frac{w_i(t)}{w(t)}. \tag{2.5}$$

In order to characterize the stable distribution of wealth, it is necessary to consider the dynamics of household wealth by rank. To accomplish this, we introduce notation for household rank based on total wealth holdings. For $k = 1, \ldots, N$, let $w_{(k)}(t)$ represent the wealth held by the k-th wealthiest household in the economy at time t, so that

$$\max(w_1(t), \ldots, w_N(t)) = w_{(1)}(t) \geq w_{(2)}(t) \geq \cdots \geq w_{(N)}(t) = \min(w_1(t), \ldots, w_N(t)). \tag{2.6}$$

Next, let $\theta_{(k)}(t)$ be the share of total wealth held by the k-th wealthiest household at time t, so that

$$\theta_{(k)}(t) = \frac{w_{(k)}(t)}{w(t)}, \tag{2.7}$$

for $k = 1, \ldots, N$.

The next step is to describe the dynamics of the ranked household wealth processes $w_{(k)}$ and ranked wealth share processes $\theta_{(k)}$, $k = 1, \ldots, N$. It is necessary to introduce the notion of a local time in order to describe these dynamics.[6] For any continuous process x, the *local time* at 0 for x is the process Λ_x defined by

$$\Lambda_x(t) = \frac{1}{2}\left(|x(t)| - |x(0)| - \int_0^t \operatorname{sgn}(x(s))\, dx(s)\right). \tag{2.8}$$

As detailed by Karatzas and Shreve (1991), the local time for x measures the amount of time the process x spends near zero. To be able to link household rank to household index, let p_t be the random permutation of $\{1, \dots, N\}$ such that for $1 \leq i, k \leq N$,

$$p_t(k) = i \quad \text{if} \quad w_{(k)}(t) = w_i(t). \tag{2.9}$$

This definition implies that $p_t(k) = i$ whenever household i is the k-th wealthiest household in the economy.

It is not difficult to show that for all $k = 1, \dots, N$, the dynamics of the ranked wealth processes $w_{(k)}$ and ranked wealth share processes $\theta_{(k)}$ are given by[7,8]

$$d\log w_{(k)}(t) = d\log w_{p_t(k)}(t) + \frac{1}{2} d\Lambda_{\log w_{(k)} - \log w_{(k+1)}}(t) - \frac{1}{2} d\Lambda_{\log w_{(k-1)} - \log w_{(k)}}(t), \quad \text{a.s.}, \tag{2.10}$$

and

$$d\log \theta_{(k)}(t) = d\log \theta_{p_t(k)}(t) + \frac{1}{2} d\Lambda_{\log \theta_{(k)} - \log \theta_{(k+1)}}(t) - \frac{1}{2} d\Lambda_{\log \theta_{(k-1)} - \log \theta_{(k)}}(t), \quad \text{a.s.} \tag{2.11}$$

A formal derivation of these equations together with a brief discussion of their intuition is provided by Fernholz (2016). Using equations (2.1) and (2.3) and the definition of $\theta_i(t)$, we have that for all $i = 1, \dots, N$,

$$d\log \theta_i(t) = \mu_i(t)\, dt + \sum_{z=1}^{M} \delta_{iz}(t)\, dB_z(t) - \mu(t)\, dt - \sum_{i=1}^{N} \sum_{z=1}^{M} \theta_i(t) \delta_{iz}(t)\, dB_z(t). \tag{2.12}$$

[6]Note that we cannot simply apply Itô's Lemma in this case since the max and min functions from equation (2.6) are not differentiable.

[7]We shall use the convention that $\Lambda_{\log w_{(0)} - \log w_{(1)}}(t) = \Lambda_{\log w_{(N)} - \log w_{(N+1)}}(t) = 0$ throughout this paper.

[8]For brevity, we write $dx_{p_t(k)}(t)$ to refer to the process $\sum_{i=1}^{N} 1_{\{i = p_t(k)\}} dx_i(t)$ throughout this paper.

According to equation (2.11), then, it follows that

$$\begin{aligned} d\log\theta_{(k)}(t) = \big(\mu_{p_t(k)}(t) - \mu(t)\big)\,dt + \sum_{z=1}^{M}\delta_{p_t(k)z}(t)\,dB_z(t) - \sum_{i=1}^{N}\sum_{z=1}^{M}\theta_i(t)\delta_{iz}(t)\,dB_z(t) \\ + \frac{1}{2}d\Lambda_{\log\theta_{(k)}-\log\theta_{(k+1)}}(t) - \frac{1}{2}d\Lambda_{\log\theta_{(k-1)}-\log\theta_{(k)}}(t), \end{aligned} \tag{2.13}$$

a.s, for all $k = 1, \ldots, N$. Equation (2.13), in turn, implies that the process $\log\theta_{(k)} - \log\theta_{(k+1)}$ satisfies, a.s., for all $k = 1, \ldots, N-1$,

$$\begin{aligned} d\left(\log\theta_{(k)}(t) - \log\theta_{(k+1)}(t)\right) = \big(\mu_{p_t(k)}(t) - \mu_{p_t(k+1)}(t)\big)\,dt + d\Lambda_{\log\theta_{(k)}-\log\theta_{(k+1)}}(t) \\ - \frac{1}{2}d\Lambda_{\log\theta_{(k-1)}-\log\theta_{(k)}}(t) - \frac{1}{2}d\Lambda_{\log\theta_{(k+1)}-\log\theta_{(k+2)}}(t) \\ + \sum_{z=1}^{M}\big(\delta_{p_t(k)z}(t) - \delta_{p_t(k+1)z}(t)\big)\,dB_z(t). \end{aligned} \tag{2.14}$$

Let α_k equal the time-averaged limit of the expected growth rate of wealth for the k-th wealthiest household relative to the expected growth rate of total wealth in the economy, so that

$$\alpha_k = \lim_{T\to\infty}\frac{1}{T}\int_0^T \big(\mu_{p_t(k)}(t) - \mu(t)\big)\,dt, \tag{2.15}$$

for $k = 1, \ldots, N$. The relative growth rates α_k are a rough measure of the rate at which household wealth reverts to the mean. In a similar manner, we wish to define the time-averaged limit of the volatility of the process $\log\theta_{(k)} - \log\theta_{(k+1)}$, which measures the relative wealth holdings of adjacent households in the distribution of wealth. For all $k = 1, \ldots, N-1$, let σ_k be given by

$$\sigma_k^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{z=1}^{M}\big(\delta_{p_t(k)z}(t) - \delta_{p_t(k+1)z}(t)\big)^2\,dt. \tag{2.16}$$

The relative growth rates α_k together with the volatilities σ_k entirely determine the shape of the stable distribution of wealth in this economy. Finally, for all $k = 1, \ldots, N$, let

$$\kappa_k = \lim_{T\to\infty}\frac{1}{T}\Lambda_{\log\theta_{(k)}-\log\theta_{(k+1)}}(T). \tag{2.17}$$

Let $\kappa_0 = 0$, as well. As shown by Fernholz (2016), the parameters α_k and κ_k are related by

$\alpha_k - \alpha_{k+1} = \frac{1}{2}\kappa_{k-1} - \kappa_k + \frac{1}{2}\kappa_{k+1}$, for all $k = 1, \ldots, N-1$. Note that we assume that the limits in equations (2.15)-(2.17) exist.

The distribution of wealth in this economy is *stable* if the limits in equations (2.15)-(2.17) all exist and if the limits in equations (2.16)-(2.17) are positive constants. In this paper, we shall only consider and apply this section's results to economies that are stable.[9] It is important to emphasize that stability of the distribution of wealth is a weaker condition than stationarity of the distribution of wealth.[10]

The *stable version* of the process $\log\theta_{(k)} - \log\theta_{(k+1)}$ is the process $\log\hat{\theta}_{(k)} - \log\hat{\theta}_{(k+1)}$ defined by

$$d\left(\log\hat{\theta}_{(k)}(t) - \log\hat{\theta}_{(k+1)}(t)\right) = -\kappa_k\,dt + d\Lambda_{\log\hat{\theta}_{(k)}-\log\hat{\theta}_{(k+1)}}(t) + \sigma_k\,dB(t), \tag{2.18}$$

for all $k = 1, \ldots, N-1$.[11] The stable version of $\log\theta_{(k)} - \log\theta_{(k+1)}$ replaces all of the processes from the right-hand side of equation (2.14) with their time-averaged limits, with the exception of the local time process $\Lambda_{\log\theta_{(k)}-\log\theta_{(k+1)}}$. By considering the stable version of these relative wealth holdings processes, we are able to obtain a simple characterization of the distribution of wealth.

Theorem 2.1. *There is a stable distribution of wealth in this economy if and only if* $\alpha_1 + \cdots + \alpha_k < 0$, *for* $k = 1, \ldots, N-1$. *Furthermore, if there is a stable distribution of wealth, then for* $k = 1, \ldots, N-1$, *this distribution satisfies*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\log\hat{\theta}_{(k)}(t) - \log\hat{\theta}_{(k+1)}(t)\right)\,dt = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}, \quad \text{a.s.} \tag{2.19}$$

We refer the reader to Fernholz (2016) for a proof of Theorem 2.1.[12] Despite this section's general, nonparametric setup, this theorem provides an analytic rank-by-rank characterization of the entire stable distribution of wealth. The theorem implies that practically all theoretical models of wealth distribution can be solved in full detail by measuring the rank-based relative growth rates α_k and volatilities σ_k of those models. Both parameters vary

[9] In fact, the setup we consider in this paper yields a stationary wealth distribution, which is a stronger condition than stability.

[10] This section's methods can also be applied to some unstable economic scenarios (see Fernholz, 2016).

[11] For each $k = 1, \ldots, N$, equation (2.18) implicitly defines another Brownian motion $B(t)$, $t \in [0, \infty)$. These Brownian motions can covary in any way across different different k.

[12] We also refer the reader to Fernholz and Karatzas (2009) for a more detailed discussion of the mathematics behind Theorem 2.1 and rank-based stochastic processes more generally.

across different ranks in the distribution, thus going beyond simpler formulations based on the equal growth rates imposed by Gibrat's law (Gabaix, 1999, 2009).

In the theoretical model we present in the next section, we measure exactly these parameters and then apply Theorem 2.1 to describe the equilibrium distribution of wealth. As we demonstrate, the extra detail and tractability that this solution technique provides allows us to also describe economic mobility and the implications of risk-sharing subgroups of households.

3 Model

Consider an economy that is populated by $N > 1$ dynastic households. Each household $i = 1, \ldots, N$ lives for a time of length S, and then at the end of its life passes on any remaining wealth to a single offspring who starts its life at that instant in time. Households have joy-of-giving bequest motives, and at each point in time $t \leq S$ they solve savings-consumption problems that determine how much to consume and how to invest their savings.

3.1 Consumption, Investment, Taxes, and Wealth

The households in this setup have two investment options: a risk-free asset available to all households that pays a return of r, and an individual-specific asset that is subject to idiosyncratic risk. For all $i = 1, \ldots, N$, we assume that the price of household i's individual-specific risky asset P_i evolves according to the stochastic differential equation

$$dP_i(t) = \lambda P_i(t)\, dt + \kappa P_i(t)\, dB_i(t), \tag{3.1}$$

where $\lambda > r$ and $\kappa > 0$ represent, respectively, the expected instantaneous return of this risky asset and the standard deviation of this instantaneous return. Note that λ and κ are the same for all households. Following the heterogeneous-agent macroeconomics literature, we assume that markets are incomplete and hence the risk involved in these N individual-specific assets cannot be shared or pooled across households.[13] As a consequence, each household in the economy faces uninsurable idiosyncratic investment risk.

We assume that the returns of the risk-free asset and the individual-specific risky assets are exogenous, a simplification that follows Benhabib et al. (2011, 2014) and others in this literature. It is straightforward, however, to endogenously derive individual-specific asset returns that evolve according to geometric Brownian motions in a general equilibrium setting. Indeed, Angeletos and Calvet (2006) and Angeletos and Panousi (2011) derive individual-specific returns to capital that are identical to equation (3.1) above, with λ equal to the expected return to capital invested in a profit-maximizing private business that has a neoclassical production function and is subject to idiosyncratic productivity shocks. In this general equilibrium model, households earn labor income, and the risk-free return r is endogenous and adjusts so that aggregate demand for the risk-free asset is zero. It follows, then, that our decision to simplify the model and make asset returns exogenous is without loss of generality and reflects this paper's focus on the distribution of wealth and economic mobility rather than asset pricing. Furthermore, because our results do not depend on the specific returns of any assets, these results should be consistent with asset returns other than the geometric Brownian motions derived by Angeletos and Panousi (2011).[14]

At each point in time $0 \leq t \leq S$ during its lifetime, each household i chooses a fraction $\phi_i(t)$ of its wealth $w_i(t)$ to invest in the risky asset and a quantity $c_i(t)$ to consume. Because households are symmetric across generations, these optimal choices do not depend on the birth time of a particular household. We assume that households obtain utility from consumption and a joy-of-giving bequest motive, and that they have utility functions that feature constant relative risk aversion (CRRA). Thus, each household $i = 1, \ldots, N$ solves

[13]For simplicity, this model abstracts from any aggregate risk or covariance of idiosyncratic risk across households, even though our solution technique as described in Section 2 can be applied to these more complex settings (see Fernholz, 2016 for a discussion). Implicitly, then, we assume that the idiosyncratic risk of the N households in this model cancels out in the aggregate.

[14]An open question for future research is whether or not the increased tractability and detail offered by this paper's approach and solution techniques may help to solve general equilibrium models with idiosyncratic investment risk that extend the setups of Angeletos and Calvet (2006), Angeletos (2007), and Angeletos and Panousi (2011).

the maximization problem

$$
\begin{aligned}
J(w,t) = \max_{c_i(t),\phi_i(t)} \; & E_t\left[\int_t^S e^{-\rho(s-t)}\frac{c_i^{1-\gamma}(s)}{1-\gamma}\,ds + e^{-\rho(S-t)}\chi\frac{\big((1-\tau)w_i(S)\big)^{1-\gamma}}{1-\gamma}\right] \\
\text{s.t. } \; & dw_i(s) = \big[rw_i(s) + (\lambda-r)\phi_i(s)w_i(s) - c_i(s)\big]\,ds + \kappa\phi_i(s)w_i(s)\,dB_i(s),
\end{aligned}
\tag{3.2}
$$

where $\gamma \geq 1$ is the coefficient of relative risk aversion, $\rho > 0$ is the discount rate, $\chi > 0$ is the intensity of each household's bequest motive, and $0 < \tau < 1$ is the economy-wide estate tax rate imposed on bequests to offspring. The symmetry of this setup ensures that in equilibrium all of the N households in the economy have the same functional form for risky-asset demand $\phi_i(t)$ and consumption $c_i(t)$. The only factors that distinguish these choices across households are their different levels of wealth.

Proposition 3.1. *For each household $i = 1,\dots,N$ and at each point in time $0 \leq t \leq S$, the policy functions $\phi_i(t)$ and $c_i(t)$ are given by*

$$
\phi_i(t) = \frac{\lambda - r}{\gamma\kappa^2}, \tag{3.3}
$$

$$
c_i(t) = \left(\frac{e^{\eta(S-t)}-1}{\eta} + \big(\chi(1-\tau)^{1-\gamma}\big)^{\frac{1}{\gamma}}e^{\eta(S-t)}\right)^{-1} w_i(t), \tag{3.4}
$$

where

$$
\eta = \frac{(1-\gamma)r-\rho}{\gamma} + \frac{(1-\gamma)(\lambda-r)^2}{2\gamma^2\kappa^2}. \tag{3.5}
$$

The proof of Proposition 3.1 can be found in Appendix A. According to the proposition, each household invests a quantity of wealth in its individual-specific risky asset that is proportional to and increasing in the expected instantaneous excess return of that asset $\lambda - r$, and that is decreasing in both risk aversion γ, and the variance of the risky asset's instantaneous return κ^2. The intuition behind these components of risky-asset demand is standard in macroeconomics and finance. The households' optimal choice of consumption varies over the life cycle and is decreasing in the intensity of the bequest motive χ and increasing in the estate tax rate τ.

The setup of this section's model is purposely parsimonious, and shares much in common with the setup of Benhabib et al. (2014) as well as the setup in Appendix B of Fernholz and Fernholz (2014). We choose this setup because of its simplicity and its ability to accurately match the U.S. wealth distribution. It is important to emphasize, however, that our general, nonparametric solution technique—described in Section 2 above—can be applied to essentially any theoretical model of wealth distribution. All that is necessary to do this are estimates of the rank-based relative growth rates α_k and volatilities σ_k generated by different models. These estimates are then combined as in equation (2.19) from Theorem 2.1 to yield a rank-by-rank characterization of the stable distribution of wealth. Furthermore, by applying our solution technique, we are also able to describe economic mobility and the distributional implications of risk-sharing subgroups in the economy.

One advantage of the simple setup of this model is that the optimization problem (3.2) is easily solved. This, in turn, is a direct consequence of assuming that households have CRRA preferences and that returns are constant over time. In general, portfolio optimization problems that combine more complex but realistic preferences with time-varying returns present difficult challenges.[15] Any progress solving these challenging problems would likely lead to promising extensions of this setup.

The next step in our analysis is to characterize the dynamics of wealth for the households. According to Proposition 3.1, for each household $i = 1, \ldots, N$, these dynamics are given by

$$\begin{aligned} dw_i(t) = &\left[r + \frac{(\lambda - r)^2}{\gamma\kappa^2} - \left(\frac{e^{\eta(S-t)} - 1}{\eta} + (\chi(1-\tau)^{1-\gamma})^{\frac{1}{\gamma}} e^{\eta(S-t)} \right)^{-1} \right] w_i(t)\, dt \\ &+ \left(\frac{\lambda - r}{\gamma\kappa} \right) w_i(t)\, dB_i(t), \end{aligned} \tag{3.6}$$

where $0 \leq t \leq S$. By Itô's Lemma, then, it follows that

$$d \log w_i(t) = \psi(t)\, dt + \left(\frac{\lambda - r}{\gamma\kappa} \right) dB_i(t), \tag{3.7}$$

where $\psi(t)$ is given by

$$\psi(t) = r + \frac{(2\gamma - 1)(\lambda - r)^2}{2\gamma^2\kappa^2} - \left(\frac{e^{\eta(S-t)} - 1}{\eta} + (\chi(1-\tau)^{1-\gamma})^{\frac{1}{\gamma}} e^{\eta(S-t)} \right)^{-1}. \tag{3.8}$$

In addition to these within-life wealth dynamics, each household i leaves an after-tax bequest of $(1 - \tau)w_i(S)$ to its newborn offspring at the end of its life.

The model as it is set up so far does not generate a stable distribution of wealth. This is

[15] For a detailed discussion of some of these difficulties and the techniques that can be used to get around them, see Campbell and Viceira (2002).

apparent from the stability condition of Theorem 2.1, which states that there exists a stable distribution of wealth if and only if the relative growth rates from equation (2.15) satisfy $\alpha_1 + \cdots + \alpha_k < 0$, for all $k = 1, \ldots, N-1$. The setup of the model so far implies that the growth rates of wealth for all dynastic households in the economy are equal at all times, which violates this stability condition. To address this issue, we introduce a redistributive fiscal policy in which the government provides newborn households with a single lump sum transfer. This transfer is financed by the estate tax τ that is imposed on all bequests. In terms of the model, we assume that each household i born at time $t \in \{S, 2S, \ldots\}$ receives a transfer equal to $\nu_i(t)$, where $0 \leq \nu_i(t) \leq \frac{\tau w(t)}{N}$, for all $i = 1, \ldots, N$.[16] Similarly, for all $k = 1, \ldots, N$, let $\nu_{(k)}(t)$ denote the transfer paid to the k-th wealthiest household at time t. For now, we impose almost no structure on these transfers and allow them to depend on many different factors including household wealth and rank. We shall specify below exactly what conditions on the transfers $\nu_i(t)$ are necessary to ensure that a stable distribution of wealth exists.

We wish to characterize the total wealth of the dynastic households in the economy. In order to accomplish this, we use equation (3.7) and link households across generations using the fact that after-tax bequests are equal to $(1-\tau)w_i(S)$. For all $i = 1, \ldots, N$, the total wealth held by dynastic household i at time $t \geq 0$ is given by

$$\begin{aligned} \log w_i(t) = \log w_i(0) &+ \sum_{j=1}^{\bar{t}(t)} \log\left(1 - \tau + \frac{\nu_i(jS)}{w_i(jS)}\right) \\ &+ \bar{t}(t)\int_0^S \psi(s)\,ds + \int_0^{t-\bar{t}(t)S} \psi(s)\,ds + \left(\frac{\lambda - r}{\gamma\kappa}\right) B_i(t). \end{aligned} \tag{3.9}$$

where

$$\bar{t}(t) = \left\lfloor \frac{t}{S} \right\rfloor, \tag{3.10}$$

denotes the largest integer less than or equal to t/S (the floor function).

The issue of stability of the distribution of wealth in models in which households face idiosyncratic investment risk can be addressed in several different ways. Our approach is most similar to Benhabib et al. (2014), who also achieve stability by introducing a redistributive transfer policy paid to newborn households. A major advantage of this approach is its

[16] This latter condition ensures that the government budget is either balanced or in surplus.

simplicity and tractability.

Alternatively, the fiscal transfer in this model can also be interpreted as the time-discounted sum of future labor income, or human wealth. In this sense, our setup is very similar to Zhu (2010), Benhabib et al. (2011), and Appendix B of Fernholz and Fernholz (2014), all of who generate stability through a combination of labor income and limited inter-generational transfers that ensure that the ratio $\frac{\nu_i(jS)}{w_i(jS)}$ from equation (3.9) does not decrease to zero. Another technique to ensure stability is to include direct frictions that prevent household wealth from falling below some positive threshold, as Champernowne (1953) and Gabaix (2009) do. In all cases, however, some mechanism must be inserted into the model to ensure that there is mean reversion and that the stability condition from Theorem 2.1 is satisfied.

3.2 The Equilibrium Distribution of Wealth

In order to use the analysis and results from Section 2 to solve this section's model, it is necessary to modify the dynastic wealth accumulation processes for the households so that they are continuous. This modification is not difficult and does not alter the steady-state equilibrium distribution of wealth in any meaningful way.

For some $t_0 > 0$, let $Z = \cup_{j=1}^{\infty}[jS, jS + t_0]$. For all $i = 1, \ldots, N$, let the modified version of the dynastic wealth process $\tilde{w}_i$ follow the stochastic differential equation

$$d\log\tilde{w}_i(t) = \psi\left(t - \bar{t}(t)S\right)\,dt + \frac{1_Z(t)}{t_0}\log\left(1 - \tau + \frac{\nu_i\left(\bar{t}(t)S\right)}{\tilde{w}_i\left(\bar{t}(t)S\right)}\right)dt + \left(\frac{\lambda - r}{\gamma\kappa}\right)dB_i(t). \quad (3.11)$$

The processes $\tilde{w}_i$ modify the corresponding dynastic household wealth processes w_i by changing the discontinuous jumps in wealth that occur from both bequest taxation at death and lump sum transfers at birth into rapid but continuous changes in wealth that occur over brief intervals of length t_0. By adjusting the length of these intervals, the steady-state distribution of the modified dynastic household wealth processes $\tilde{w}_i$ approximates the steady-state distribution of the unchanged dynastic household wealth processes w_i arbitrarily well.

As intended, the modified wealth processes $\tilde{w}_i$ are of the same form as equation (2.1) from Section 2. Indeed, equation (2.1) is equivalent to equation (3.11) if, for all $i = 1, \ldots, N$, we set

$$\mu_i(t) = \psi\left(t - \bar{t}(t)S\right) + \frac{1_Z(t)}{t_0}\log\left(1 - \tau + \frac{\nu_i\left(\bar{t}(t)S\right)}{\tilde{w}_i\left(\bar{t}(t)S\right)}\right), \quad (3.12)$$

and

$$\sum_{z=1}^{M} \delta_{iz}(t)\, dB_z(t) = \left(\frac{\lambda - r}{\gamma \kappa} \right) dB_i(t). \tag{3.13}$$

Having established the values of the general model from the previous section, we can now apply Theorem 2.1 to solve for the stable distribution of wealth of this section's model.

For all $i = 1, \dots, N$, let the modified wealth shares processes be defined by $\tilde{\theta}_i = \frac{\tilde{w}_i}{\tilde{w}}$, where $\tilde{w} = \tilde{w}_1 + \cdots + \tilde{w}_N$. Similarly, for all $k = 1, \dots, N$, define the parameters $\tilde{\alpha}_k$ and $\tilde{\sigma}_k$ by

$$\tilde{\alpha}_k = \lim_{j \to \infty} \frac{1}{S} \log \left(1 - \tau + \frac{\nu_{(k)}(jS)}{\tilde{w}_{(k)}(jS)} \right) - \frac{1}{SN} \sum_{\ell=1}^{N} \log \left(1 - \tau + \frac{\nu_{(\ell)}(jS)}{\tilde{w}_{(\ell)}(jS)} \right), \tag{3.14}$$

$$\tilde{\sigma}_k = \sqrt{2} \left(\frac{\lambda - r}{\gamma \kappa} \right). \tag{3.15}$$

These parameters correspond to the time-averaged rank-based relative growth rates and idiosyncratic volatilities of this theoretical model, as defined more generally by equations (2.15) and (2.16) above. We demonstrate this in the proof of Proposition 3.2 in Appendix A. Because $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \cdots = \tilde{\sigma}_N$, for simplicity we denote this common volatility by $\tilde{\sigma} = \tilde{\sigma}_1$.

As in Section 2, we wish to consider stable versions of the relative wealth processes $\log \tilde{\theta}_{(k)} - \log \tilde{\theta}_{(k+1)}$ and then characterize the equilibrium distribution of wealth using those stable processes. In this case, however, because we have a model with an explicit parametric structure, it also possible to describe the corresponding stable versions of the rank wealth processes $\tilde{w}_{(k)}$. Following equation (2.18), these stable versions should smooth fluctuations in the growth rate of the original processes by replacing these growth rates with their time-averaged limits. For all $k = 1, \dots, N-1$, let the stable version of the rank wealth processes $\tilde{w}_{(k)}$ be the processes $w^*_{(k)}$ defined by

$$\begin{aligned} d \log w^*_{(k)}(t) = (\tilde{\alpha}_k + a)\, dt + \frac{\tilde{\sigma}}{\sqrt{2}}\, dB_{p_t(k)}(t) \\ + \frac{1}{2} d\Lambda_{\log w^*_{(k)} - \log w^*_{(k+1)}}(t) - \frac{1}{2} d\Lambda_{\log w^*_{(k-1)} - \log w^*_{(k)}}(t), \end{aligned} \tag{3.16}$$

where a is a constant that measures the common growth rate of total wealth in the economy.[17] These stable versions of the rank wealth processes yield corresponding stable versions of the

[17]As we demonstrate below, this common growth rate has no effect on the equilibrium distribution of wealth, economic mobility, or the implications of risk-sharing subgroups.

rank wealth share processes $\theta^*_{(k)}$ using the definition $\theta^*_{(k)} = \frac{w^*_{(k)}}{w^*}$, where $w^* = w^*_{(1)} + \cdots + w^*_{(N)}$. It is straightforward to show that the stable versions $\theta^*_{(k)}$ generate stable versions of the processes $\log\tilde{\theta}_{(k)} - \log\tilde{\theta}_{(k+1)}$ that match the original definition from equation (2.18) in Section 2.

Proposition 3.2. *There exists a steady-state distribution of wealth in this economy if and only if $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k < 0$, for $k = 1, \ldots, N-1$. Furthermore, if there is a steady-state distribution of wealth, then for $k = 1, \ldots, N-1$, this distribution satisfies*

$$E\left[\log\theta^*_{(k)}(t) - \log\theta^*_{(k+1)}(t)\right] = \frac{\tilde{\sigma}^2}{-4(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)}, \quad \text{a.s.} \tag{3.17}$$

Proposition 3.2 analytically characterizes the entire shape of the equilibrium distribution of wealth in this economy with idiosyncratic investment risk.[18] The bulk of the proof of this proposition is simply an application of Theorem 2.1 (the full proof is in Appendix A). The other main step in the proof is to show that the parameters $\tilde{\alpha}_k$ and $\tilde{\sigma}$ from equations (3.14) and (3.15) correspond to the rank-based relative growth rates and volatilities as defined by equations (2.15) and (2.16) above.

The $N-1$ equations from Proposition 3.2 together with the fact that $\theta^*_{(1)}(t) + \cdots + \theta^*_{(N)}(t) = 1$ yields a simple system of equations that describes what share of the economy's total wealth each household will hold on average over time in this stable distribution.[19] Consider an important special case of this proposition. If the values $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k$ are equal for all $k = 1, \ldots, N-1$, then the stable equilibrium distribution of wealth in this setup is approximately Pareto with parameter $-4\tilde{\alpha}_1/\tilde{\sigma}^2 > 0$. According to Proposition 3.2, then, this model generates a Pareto distribution of wealth only in some cases. In fact, the full equilibrium describes a more general distribution of wealth in which the Pareto parameter may vary across different households. According to Kennickell (2009), Davies et al. (2011), and Saez and Zucman (2014), a general distribution in which the Pareto parameter varies in this way better describes the empirical distributions of wealth in most countries.

Proposition 3.2 states that two factors alone describe the stable distribution of wealth, just like in equation (1.1) above. The first factor is household exposure to idiosyncratic

[18]Note that this theorem can be extended to a setup in which households have different expected returns on their individual-specific investments. See Ichiba et al. (2011) for the technical details.

[19]In fact, this system of equations yields a solution for the certainty-equivalent approximation of the wealth shares $\theta^*_{(1)}(t), \ldots, \theta^*_{(N)}(t)$, rather than the time-averaged value of those shares. See Banner et al. (2005) for a discussion.

investment risk, $\tilde{\sigma}$, and the second factor is the reversion rates of household wealth, measured by $-\tilde{\alpha}_k$. An increase in households' exposure to uninsurable investment risk linearly increases the extent of inequality in the economy. This follows because for any $k = 1, \ldots, N-1$, a higher value of $\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)$ implies a larger gap between the wealth holdings of two different households. Conversely, an increase in cross-sectional mean reversion (as measured by the reversion rates) decreases the extent of inequality in the economy. The effects of any change in policy or the economic environment on the equilibrium distribution of wealth are entirely determined by the effects of this change on these two shaping factors.

According to equation (3.15), household exposure to idiosyncratic investment risk in equilibrium is increasing in the excess return of households' risky individual-specific assets, $\lambda - r$, and decreasing in both risk aversion, γ, and the standard deviation of the return of these risky individual-specific assets, κ. According to equation (3.14), the reversion rates of household wealth depend on how the ratios of lump sum transfers (paid to newborn households) relative to wealth holdings, $\frac{\nu_{(k)}(t)}{\tilde{w}_{(k)}(t)}$, vary across different ranked households. In Section 4, we parameterize this model and impose more structure on these lump sum transfers. As Proposition 3.2 demonstrates, this structure is an important factor in shaping the equilibrium distribution of wealth.

Proposition 3.2 also states that asymptotic stability in this economy requires that the expected growth rate of wealth for poor households be higher than for wealthy households. This is ensured by the restriction that $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k < 0$, for $k = 1, \ldots, N-1$. In other words, stability requires positive reversion rates for the wealthiest households in the economy—their wealth holdings must revert to the mean over time. What happens if all households have the same expected growth rates of wealth? In this case, there is no asymptotic stability in the distribution of wealth, and instead wealth becomes increasingly concentrated at the top over time.

Proposition 3.3. *If the relative growth rates are such that $\tilde{\alpha}_1 = \tilde{\alpha}_2 = \cdots = \tilde{\alpha}_N$, then the share of the economy's total wealth held by the wealthiest single household, $\tilde{\theta}_1$, satisfies*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{\theta}_{(1)}(t)\, dt = 1, \quad \text{a.s.} \tag{3.18}$$

Proposition 3.3 describes how wealth becomes increasingly concentrated over time if all households have the same expected growth rates of wealth, a result that is almost identical to the main result of Fernholz and Fernholz (2014).[20] Because the only factor that distinguishes the relative growth rates $\tilde{\alpha}_k$ across households in this model are the lump sum transfers that newborn households receive, $\nu_i(t)$, this result confirms that this policy is essential to ensuring that a stable distribution of wealth exists in this economy. Indeed, only if the lump sum transfers proportionally benefit poor households more than wealthy households does a stable distribution of wealth exist. This restriction on proportional benefits is equivalent to $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k < 0$, for $k = 1, \ldots, N-1$. Finally, Proposition 3.3 describes a specific example of divergence, but this result can be generalized to describe many other instances in which a divergent subpopulation of households eventually holds all wealth.[21]

3.3 Economic Mobility

One of the advantages of our solution technique in this paper is that it allows for a simple analytic characterization of the extent of mobility that exists in the economy. In fact, we are able to establish a direct link between economic mobility, inequality, and cross-sectional mean reversion.

In order to explore the issue of mobility, it is useful to examine the process $\log \theta^*_{(k)} - \log \theta^*_{(k+1)} \geq 0$, which describes the dynamics of the difference between the shares of total wealth in the economy held by the k-th and $k+1$-th wealthiest households. The expected time it takes for this process to reach zero is of particular interest. Indeed, whenever $\log \theta^*_{(k)} - \log \theta^*_{(k+1)}$ reaches zero, one household has overtaken another household in terms of wealth holdings—one household has improved its economic standing, which is the meaning of economic mobility. The longer it takes, on average, for one household to overtake another, the less mobility there is in the economy.

To measure economic mobility, we define the stopping time

$$S_k(t) = \inf\{s \geq 0 : \theta^*_{(k)}(t+s) = \theta^*_{(k+1)}(t+s)\}, \tag{3.19}$$

for $k = 1, \ldots, N-1$. This stopping time marks the first time after t that the $k+1$-th wealthiest household overtakes the k-th wealthiest household. Note that $S_k(t)$ is defined in terms of the steady-state versions of the processes $\theta^*_{(k)}$, since we wish to characterize the extent of economic mobility in the steady-state equilibrium of the model. The following

[20]For brevity, we refer the reader to Fernholz and Fernholz (2014) for a proof of Proposition 3.3.
[21]See Theorem 2.4 of Fernholz (2016).

theorem describes the long-run behavior of this stopping time in two different ways.

Theorem 3.4. *If there exists a steady-state equilibrium distribution of wealth, then for all* $k = 1, \dots, N-1$,

$$E\left[S_k(t) \,\middle|\, \theta^*_{(k)}(t), \theta^*_{(k+1)}(t)\right] = \frac{\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)}{-2(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)}, \tag{3.20}$$

$$E\left[S_k(t)\right] = \frac{\tilde{\sigma}^2}{8(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)^2}. \tag{3.21}$$

Theorem 3.4 provides a direct measure of the extent of economic mobility in this setup. The first result, equation (3.20), states that after controlling for the extent of inequality in the distribution of wealth, long-run economic mobility is determined by the reversion rates of household wealth. In particular, this equation shows that in the steady state, the expected time before the $k+1$-th wealthiest household overtakes the k-th wealthiest household is proportional to the sum of reversion rates for different ranked households, $-(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)$, and the initial inequality between the two households, $\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)$. The second result, equation (3.21), extends this result, stating that long-run economic mobility is entirely determined by household exposure to idiosyncratic investment risk and mean reversion.

Theorem 3.4 represents one of the first attempts to describe in closed form the determinants of mobility in a macroeconomic model in which households face uninsurable risks to their incomes. The determinants of the extent of mobility—household exposure to idiosyncratic investment risk and the reversion rates of household wealth—connect directly with the discussion of the equilibrium distribution of wealth in Section 3.2. These determinants of both inequality and mobility are themselves determined by the structural parameters of the model, with the structure of lump sum transfers to newborn households the main determinant of mean reversion, and risk aversion and the excess return and volatility of households' individual-specific investments the main determinants of exposure to idiosyncratic investment risk.

Interestingly, Theorem 3.4 shows that of the two factors that shape the distribution of wealth in this setup, cross-sectional mean reversion is the more important determinant of economic mobility. The unevenness of this result is perhaps surprising. After all, there is no obvious reason why this one factor should have a disproportionate effect on economic mobility.

The characterization of economic mobility in this setup provides several interpretations of some recent empirical results, many of which have garnered attention from the public. For example, Chetty et al. (2014) examine data from anonymous earnings records and find substantial variation in intergenerational mobility across different geographical areas of the United States. The authors show that those areas with more intergenerational mobility also tend to have more unequal income distributions. This result is broadly consistent with the predictions of our model, as are the empirical results of Corak (2013), who shows that those countries with more unequal income distributions also tend to have lower mobility. More generally, Theorem 3.4 provides a broad interpretation of the geographical variation in intergenerational mobility that both Corak (2013) and Chetty et al. (2014) document. Because the theorem directly connects economic mobility to inequality and mean reversion, the implication is that any geographical variation in mobility also reflects geographical variation in inequality and mean reversion. Furthermore, after controlling for inequality, any remaining geographic variation in mobility reflects a similar geographic variation in mean reversion.

Theorem 3.4 provides results about one particular measure of economic mobility, but it is natural to wonder about other more common measures of mobility as well. Later in the paper, when we consider several different parameterizations and solve the model numerically, we shall examine the model's predictions for measures of mobility that are common in empirical work. It is also possible to extend our theoretical results to more general measures of mobility.

Consider the process $w^{**}_{(k)}$, $k = 1, \dots N$, given by

$$\begin{aligned} d\log w^{**}_{(k)}(t) = x(\tilde{\alpha}_k + a)\,dt + \frac{\sqrt{x}\tilde{\sigma}}{\sqrt{2}}\,dB_{p_t(k)}(t) \\ + \frac{1}{2}d\Lambda_{\log w^{**}_{(k)} - \log w^{**}_{(k+1)}}(t) - \frac{1}{2}d\Lambda_{\log w^{**}_{(k-1)} - \log w^{**}_{(k)}}(t), \end{aligned} \tag{3.22}$$

where x is a positive constant. Similarly, for all $k = 1, \dots N$, let $\theta^{**}_{(k)} = \frac{w^{**}_{(k)}}{w^{**}}$, where $w^{**} = w^{**}_{(1)} + \cdots + w^{**}_{(k)}$. It follows that $\theta^{*}_{(k)}$ and $\theta^{**}_{(k)}$ yield the same steady-state distributions of wealth, since, for all $k = 1, \dots, N-1$,

$$E[\log\theta^{**}_{(k)}(t) - \log\theta^{**}_{(k+1)}(t)] = \frac{x\tilde{\sigma}^2}{-2x(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)} = \frac{\tilde{\sigma}^2}{-2(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)}, \tag{3.23}$$

which by Proposition 3.2, is also equal to $E[\log\theta^{*}_{(k)}(t) - \log\theta^{*}_{(k+1)}(t)]$. The processes $\theta^{**}_{(k)}$ adjust the household exposure to idiosyncratic investment risk and the reversion rates of our original model by equal proportions, thus leaving the equilibrium distribution of wealth unchanged. This adjustment does, however, affect equilibrium economic mobility. Indeed, even if inequality is unchanged, any change in reversion rates must affect mobility.

Let the stopping time $T^*_{k,l}(t)$ be defined as

$$T^*_{k,l}(t) = \inf\{s \geq 0 : \theta^*_i(t+s) = \theta^*_{(l)}(t+s) \mid \theta^*_i(t) = \theta^*_{(k)}(t)\}, \tag{3.24}$$

for $k = 1, \ldots, N$ and $l \neq k$. Furthermore, let the stopping time $T^{**}_{k,l}(t)$ be defined as in equation (3.24) but with the processes $\theta^{**}_{(k)}$ replacing the processes $\theta^*_{(k)}$, for all $k = 1, \ldots, N$. The stopping times $T^*_{k,l}(t)$ and $T^{**}_{k,l}(t)$ measure the time it takes for the k-th wealthiest household at time t to transition to becoming the l-th wealthiest household in the economy. In other words, this general measure of economic mobility captures the transition time between any two ranks in the distribution of wealth.

Theorem 3.5. *If there exists a steady-state equilibrium distribution of wealth, then for all $k = 1, \ldots, N$ and $l \neq k$,*

$$E[T^*_{k,l}(t)] = xE[T^{**}_{k,l}(t)]. \tag{3.25}$$

In many ways, Theorem 3.5 is more general than Theorem 3.4 since it describes the changing behavior of the stopping times $T^*_{k,l}(t)$, which represent much broader measures of mobility than the stopping times $S_k(t)$. Theorem 3.5 demonstrates that the result from Theorem 3.4 that mobility is increasing in the reversion rates of household wealth, after controlling for inequality, holds for more general measures of mobility as well.

3.4 Risk-Sharing Subgroups

In addition to generating results on economic mobility, our solution technique allows us to describe the implications of the presence of a subset of households that are able to share their idiosyncratic investment risk with each other. In this sense, we are able to examine different types of partially complete markets and to compare distributional and welfare outcomes in those scenarios with those in the fully incomplete market setup that we have focused on so far.

Not surprisingly, if some subset of households in our model are able to share their idiosyncratic investment risks with each other, then this subset of households can achieve a better outcome in terms of expected utility. This expansion of complete markets also must have an impact on the equilibrium distribution of wealth, since the wealth holdings of a subset of households are automatically equalized in this scenario. It is less clear, however, what the impact on inequality will be and how the rest of the households in the economy might be affected, if at all.

Using our rank-based solution technique, we are able to address these questions. We consider a scenario in which households $i = 1, \ldots, m$, $1 < m < N$, are now able to fully share with each other the idiosyncratic risk associated with their individual-specific assets.[22] Because of the symmetry across households and individual-specific assets in this setup, all households in this risk-sharing subgroup choose identical portfolios and choose to fully diversify across all available risky investments in equilibrium.

In order to characterize the distributional implications of a risk-sharing subgroup of households, we define, for $k = 1, \ldots, N - m + 1$, the occupation times

$$\zeta_k = E\left[1_{\{r_t(1)=k\}}(t)\right], \tag{3.26}$$

where

$$r_t(i) = k \quad \text{if} \quad w_i^*(t) = w_{(k)}^*(t). \tag{3.27}$$

We assume that the households in this risk-sharing subgroup are ranked by index, so that household $i = 1$ has the highest rank in the group and household $i = m$ has the lowest rank in the group. The occupation times ζ_k describe the time that each household in the risk-sharing subgroup spends in each possible rank of the distribution. Because we have ordered the rank of the households in the subgroup by index, it follows that there are only $N - m + 1$ possible ranks that each household in this subgroup can occupy. It also follows that $\zeta_1 + \cdots + \zeta_{N-m+1} = 1$.

The existence of a risk-sharing subgroup has two separate effects on the equilibrium distribution of wealth. First, the households in the subgroup are now subject to less volatility from their risky investment and this has an effect on the growth rate of their wealth holdings. Second, there is an equilibrium change in behavior—because the households in the subgroup now have better risky investment options, they alter their consumption and invest more aggressively in higher-return risky assets. These two effects are separate in the sense that even in the absence of any equilibrium change in household behavior, there is still a substantial impact on the economy from the fact that the wealth holdings of those households in the

[22]The assumption that households $i = 1, \ldots, m$ are able to risk share, rather than some other subset of households, is for simplicity and without loss of generality.

risk-sharing subgroup are now less volatile. This effect only grows larger as risk-sharing households alter their behavior in equilibrium and invest more aggressively in their pooled risky asset. We distinguish between these two separate effects in our characterization of the distributional implications of increased risk sharing.

Theorem 3.6. *Suppose that the households in the risk-sharing subgroup do not change their consumption or risky-asset allocations. If there exists a steady-state equilibrium distribution of wealth, then the occupation times ζ_k must satisfy*

$$\sum_{k=1}^{N-m+1} \zeta_k \left(\tilde{\alpha}_k + \cdots + \tilde{\alpha}_{k+m-1}\right) = \frac{(1-m)(N-m)}{4N}\tilde{\sigma}^2. \tag{3.28}$$

If no solution exists, then $\zeta_1 = 1$.

Theorem 3.6 characterizes the occupation times ζ_k and describes the distributional outcome for the risk-sharing subgroup of households in the absence of any change in their consumption or portfolio choices. The factors that determine this outcome are essentially the same factors that shape the distribution of wealth and determine economic mobility—the reversion rates, $-\tilde{\alpha}_k$, and household exposure to idiosyncratic investment risk, $\tilde{\sigma}^2$. These factors are themselves determined by the structural parameters of the model, as discussed in Section 3.2 above.

Recall the stability condition from Proposition 3.2, which implies that

$$\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_m < 0 < \tilde{\alpha}_{N-m+1} + \cdots + \tilde{\alpha}_N. \tag{3.29}$$

Equation (3.29) can be generalized, since the sums $\tilde{\alpha}_k + \cdots + \tilde{\alpha}_{k+m-1}$ will usually be larger (and sometimes positive) for larger values of k. As a consequence, equation (3.28) implies that if cross-sectional mean reversion is close to zero (and hence the sums $\tilde{\alpha}_k + \cdots + \tilde{\alpha}_{k+m-1}$ are also close to zero) and household exposure to idiosyncratic investment risk is high, then the risk-sharing subgroup of households will spend most of its time at the top of the wealth distribution.[23] Conversely, if mean reversion is substantially above zero and household exposure to idiosyncratic investment risk is low, then the risk-sharing subgroup of households will spend more time near the bottom of the wealth distribution. Of course, if cross-sectional

[23]If $\tilde{\sigma}^2$ is sufficiently high, then there may be no solution to equation (3.28). In this case, $\zeta_1 = 1$ and the risk-sharing subset of households remains permanently at the top of the wealth distribution.

mean reversion is close to zero and household exposure to idiosyncratic investment risk is high, then there is also substantial wealth inequality in the economy, and vice versa.

This result, then, may be restated in terms of the shape of the equilibrium distribution of wealth: if inequality is high, then the risk-sharing subgroup of households rises to the top of the distribution, and if inequality is low, then the risk-sharing subgroup of households falls near the bottom of the distribution. These effects are solely a consequence of the lower volatility of household wealth for those households in the subgroup, since we assume in Theorem 3.6 that these households do not change their consumption or risky-asset allocations.

Corollary 3.7. *If there exists a steady-state equilibrium distribution of wealth, then the occupation times ζ_k must satisfy*

$$\sum_{k=1}^{N-m+1} \zeta_k \left(\tilde{\alpha}_k + \cdots + \tilde{\alpha}_{k+m-1}\right) = \frac{m(1-m)(N-m)}{4N}\tilde{\sigma}^2 - \frac{m(N-m)}{N}\Delta, \tag{3.30}$$

where

$$\Delta = \frac{(m-1)(\lambda-r)^2}{2\gamma\kappa^2} + \frac{1}{S}\log\left[\frac{\eta'\left((1+\eta\chi^{\frac{1}{\gamma}})e^{\eta S}-1\right)}{\eta\left((1+\eta'\chi^{\frac{1}{\gamma}})e^{\eta' S}-1\right)}\right], \tag{3.31}$$

and

$$\eta' = \eta + \frac{(m-1)(1-\gamma)(\lambda-r)^2}{2\gamma^2\kappa^2}. \tag{3.32}$$

If no solution exists, then $\zeta_1 = 1$.

Corollary 3.7 extends Theorem 3.6 and characterizes the occupation times ζ_k without ignoring the equilibrium changes in consumption and portfolio choice for the risk-sharing subgroup of households. The main difference between equations (3.28) and (3.30) is the term $\Delta > 0$. This term captures the higher returns on wealth in the risk-sharing subgroup, a direct consequence of those households in the subgroup investing more aggressively in the many high-return individual-specific risky assets that are now available to them. More precisely, the optimal risky-asset share $\phi_i(t)$ from equation (3.3) is higher than before for the risk-sharing households since the variance of the return on an equal-weighted investment in all of the risky assets available to them is lower than before. The higher risky-asset investment share of these households leads to higher expected returns. Other than this extra equilibrium effect, the implications of risk sharing as characterized by Theorem 3.6 and

Corollary 3.7 are essentially the same. In both cases, higher inequality makes it more likely that the risk-sharing subgroup of households will rise to the top of the wealth distribution.

Our results so far characterize only the distributional implications of a risk-sharing subgroup of households. There are, however, also effects on both welfare and the long-run expected growth rate of wealth for all households in the economy, even those that are not in the risk-sharing subgroup. In all cases, the presence of a risk-sharing subgroup of households leads to higher welfare and higher expected long-run growth rates of wealth for all households. The fact that this increase in welfare and the rate of wealth accumulation extends even to those households that are not part of the risk-sharing subgroup merits some discussion.

The main reason for this is that households in the risk-sharing subgroup achieve higher expected growth rates of wealth simply by sharing risk, even in the absence of changes in their consumption or risky-asset allocations. This is a direct consequence of risk sharing across households (see the proof of Theorem 3.6 in Appendix A). The faster pace of wealth accumulation for this subset of households then raises the rate of wealth accumulation for the entire economy, and this faster accumulation benefits all households. In our setup, this spread of faster growth is driven by the redistributive lump sum transfers provided to newborn households. These transfers raise the rate of wealth accumulation for those households at the bottom of the wealth distribution, which are more likely to be the households outside the risk-sharing subgroup. This same logic, in which higher growth rates at the bottom of the distribution are central, describes the impact of a risk-sharing subgroup of households in any model with a stable distribution of wealth. Indeed, the general stability condition for the nonparametric model in Theorem 2.1 of Section 2 implies that lower-ranked households must grow faster than higher-ranked households. If this stability condition is met, then a risk-sharing subgroup of households will always raise the rate of wealth accumulation for all households in the economy.

4 Numerical Results

In this section, we parameterize the model of Section 3 and generate numerical results for the equilibrium distribution of wealth, economic mobility, and the distributional implications of risk-sharing subgroups of households. We also explore how changes in the structural parameters of the model affect these results.

4.1 Baseline Parameterization

For all parameterizations in this section, we set the total number of households in the economy $N = 1,000,000$. The values of the basic structural parameters of the model r, γ, ρ, and χ are chosen consistent with the macroeconomics literature. We set the risk-free rate of return $r = 0.03$, the coefficient of relative risk aversion $\gamma = 1.5$, the discount rate $\rho = 0.03$, and the intensity of each household's bequest motive $\chi = 1$.

Parameterizing the returns of households' idiosyncratic investments is more difficult. Both Flavin and Yamashita (2002) and Moskowitz and Vissing-Jorgensen (2002) provide estimates of different components of these idiosyncratic returns, with the former focused on the return of housing and the latter focused on the return of private equity. These analyses led Angeletos (2007) to establish a basic calibration in which the expected investment return is approximately 7% with a standard deviation of 20%. In our baseline parameterization, we adopt these same values so that $\lambda = 0.07$ and $\kappa = 0.2$. By equation (3.15), this implies that

$$\tilde{\sigma} = \sqrt{2}\left(\frac{\lambda - r}{\gamma\kappa}\right) = 0.189. \tag{4.1}$$

We also assume that $S = 50$, so that households live and earn income for 50 years, and that the common estate tax rate $\tau = 0.2$, a value that is consistent with both the related literature (Zhu, 2010; Benhabib et al., 2011) and the average estate tax paid in the U.S.

The remaining parameters of the model, the lump sum transfers to newborn households $\nu_1, \ldots, \nu_N$, are more delicate. These transfers are essential in shaping the equilibrium distribution of wealth, as can be seen from equation (3.17) and from the expression for the parameters $\tilde{\alpha}_k$ in equation (3.14). For this baseline parameterization, we assume that all households receive the same lump sum transfer, so that $\nu_1 = \nu_2 = \cdots = \nu_N$, and we choose the common value for this transfer that most closely matches the U.S. distribution of wealth in 2012 as reported by Saez and Zucman (2014).

Let ν denote the common lump sum transfer for all newborn households, and assume that this transfer is constant over time. Furthermore, let $\bar{\nu}$ denote this common lump sum transfer as a fraction of the total wealth of all households in the economy, so that $\nu = \bar{\nu}\tilde{w}$.[24] In this case, the equilibrium expression for the parameters $\tilde{\alpha}_k$ from equation (3.14) becomes

$$\begin{aligned}\tilde{\alpha}_k &= \lim_{j\to\infty} \frac{1}{S} \log\left(1 - \tau + \frac{\nu}{\tilde{w}_{(k)}(jS)}\right) - \frac{1}{SN}\sum_{\ell=1}^{N} \log\left(1 - \tau + \frac{\nu}{\tilde{w}_{(\ell)}(jS)}\right) \\ &= \lim_{j\to\infty} \frac{1}{S} \log\left(1 - \tau + \frac{\bar{\nu}}{\tilde{\theta}_{(k)}(jS)}\right) - \frac{1}{SN}\sum_{\ell=1}^{N} \log\left(1 - \tau + \frac{\bar{\nu}}{\tilde{\theta}_{(\ell)}(jS)}\right). \qquad (4.2)\end{aligned}$$

According to equation (4.2), the reversion rates $-\tilde{\alpha}_k$ are determined primarily by the ratio of lump sum transfers to total wealth $\bar{\nu}$ and the end-of-life wealth shares $\tilde{\theta}_{(k)}(jS)$. Because the reversion rates $-\tilde{\alpha}_k$ themselves determine the steady-state equilibrium wealth shares $\theta^*_{(k)}$ according to equation (3.17), in equilibrium the steady-state shares $\theta^*_{(k)}$ solve both equations (3.17) and (4.2).[25]

Using equation (3.17), we solve for the steady-state log shares of the total wealth in the economy held by all $1,000,000$ households in the economy. Figure 1 plots these log shares against those households' log percent rank in the distribution of wealth for the baseline parameterization of the model. The value of the common lump sum transfer ratio $\bar{\nu}$ for this parameterization is $\frac{\tau}{N} = \frac{0.2}{N}$, which means that all of the government's total revenue from the estate tax is transferred back to the households. This value was chosen so that the steady-state equilibrium shares generated by the model would most closely match the 2012 U.S. wealth shares data as reported by Saez and Zucman (2014). Table 1 reports the shares of wealth held by different subsets of households in the economy for this baseline parameterization of the model and for the U.S. in 2012. As the table demonstrates, despite the simplicity of this transfer policy in which all households receive the same lump sum transfers, the baseline parameterization of the model is able to achieve a near perfect fit of

[24]Note that $\bar{\nu}$ must satisfy $0 \leq \bar{\nu} \leq \frac{\tau}{N}$, which guarantees that the government budget is either balanced or in surplus.

[25]Note that the wealth shares from equations (3.17) and (4.2) are different, since the shares from equation (3.17) are steady-state shares generated by the stable versions of the rank wealth processes $w^*_{(k)}$ as defined by equation (3.16), while the shares in equation (4.2) are end-of-life shares for the modified rank wealth processes $\tilde{w}_{(k)}$ as defined by equation (3.11). This difference is accounted for by appropriately adjusting the steady-state shares $\theta^*_{(k)}$ to generate end-of-life shares.

the data for the shares held by different subgroups within the top 1% of households in the economy. This parameterization does, however, respectively underestimate and overestimate the shares of total wealth held by the top 1-10% and bottom 90% of households in the economy.

Theorem 3.4 characterizes the expected time before the $k+1$-th wealthiest household in the economy overtakes the k-th wealthiest household. While this closed-form, analytic characterization is useful, we also wish to examine the predictions of our model for more common measures of economic mobility. We generate predictions from the model for the intergenerational rank-rank mobility measure used by Chetty et al. (2014) and Boserup et al. (2014) to examine U.S. income and Danish wealth mobility, respectively, as well as the medium-term rank correlations used by Kopczuk et al. (2010) to describe U.S. income mobility.

Figure 3 plots the starting wealth percent rank and corresponding ending wealth percent rank 50 years later for different households in the equilibrium distribution of wealth in our baseline parameterization of the model. The figure shows the results after averaging over 500 50-year simulations of the model. The regression line in the figure has a slope of 0.737 and intercept of 13.10, a result that implies a high level of persistence in wealth formation across generations of households (a steeper slope indicates less economic mobility). Indeed, Boserup et al. (2014) report a substantially flatter slope using intergenerational Danish wealth data. While some of this difference is due to the lower wealth inequality in Denmark versus the U.S., this gap in inequality is unlikely to be large enough to account for our results in Figure 3.

There are several possible explanations for this difference. First, this result is consistent with the low intergenerational mobility reported by Boserup et al. (2014), Clark (2014), and Lindahl et al. (2015) when tracking the persistence of wealth and other measures of social status across multiple (more than two) generations. One interpretation of our estimate in Figure 3 is that it lends support to these empirical studies and suggests that standard one-generation estimates of mobility are too high.

Another possibility is that the idiosyncratic volatilities of wealth from our nonparametric model in Section 2, σ_k, are underestimated in this baseline parameterization of the model of Section 3. Higher estimates of these parameters would yield correspondingly higher estimates for cross-sectional mean reversion, and this would imply higher economic mobility in equilibrium as shown by Theorem 3.5. This possibility is supported by the fact that the purely empirical estimates of the parameters σ_k constructed by Fernholz (2016) are substantially higher than our estimates for this baseline parameterization. This discrepancy is largely because our model does not include idiosyncratic labor income risk, an assumption necessary to maintain tractability and solve our general equilibrium model. Any extensions of our framework that include other such sources of idiosyncratic risk would yield higher estimates of intergenerational wealth mobility.

Table 3 reports the 10-year and 20-year rank correlations for the equilibrium distribution of wealth in our baseline parameterization. As with the intergenerational rank-rank measure of Figure 3, the results in Table 3 are obtained by averaging over 500 10-year and 20-year simulations of the model. The 10-year and 20-year rank correlations reported in the table are higher than the corresponding rank correlations reported by Kopczuk et al. (2010) using U.S. income data. Because wealth mobility tends to be lower than income mobility (see, for example, Boserup et al., 2014), some of this difference is surely because we are modeling the distribution of wealth rather than income.

In Figure 7, we show the average rank over time of a risk-sharing subgroup of 100 households that starts in the middle of the equilibrium wealth distribution for our baseline parameterization of the model. The figure shows the average outcome for this subgroup in the absence of changes in these households' decisions about consumption or risky-asset allocations. If we were to include such changes in behavior, then this risk-sharing subgroup would rise to the top of the distribution and remain there permanently. Even in the absence of these growth-enhancing changes in behavior, Figure 7 demonstrates that the subgroup quickly rises in rank until it settles around the top 10% of households. This rise within the equilibrium distribution is due entirely to the direct increase in the rate of wealth accumulation caused by risk sharing, as discussed in Section 3.4.

Figure 7 also plots the rank of an individual household over time in the absence of any risk-sharing subgroup in the economy. As the figure clearly shows, the path of each risk-sharing household through different ranks of the wealth distribution over time is much less volatile than the path of an individual household outside this subgroup. Furthermore, the households within the risk-sharing subgroup do not appear to revert back to the middle of the wealth distribution in the way that a household outside this subgroup does (see also Figure 3). Interestingly, Clark (2014) reports a similar lack of mean reversion over time among certain elite groups of people. Our results suggest that risk sharing among members of those elite groups may be an explanation for this observation.

4.2 Alternate Parameterizations: The Role of Idiosyncratic Investment Risk and Cross-Sectional Mean Reversion

One of the common themes from the theoretical results of Section 3 is that two factors—the reversion rates of household wealth and household exposure to idiosyncratic investment risk—entirely determine the equilibrium distribution of wealth and economic mobility. As shown by equations (3.17) and (3.21), an increase in mean reversion, as measured by $-\tilde{\alpha}_k$, reduces inequality, and an increase in exposure to idiosyncratic investment risk, as measured by $\tilde{\sigma}^2$, increases inequality.

These theoretical results can be observed by considering two alternate parameterizations that slightly modify the baseline parameterization. In the first alternate, the common lump sum transfer ratio $\bar{\nu}$ is decreased to $\frac{0.5\tau}{N} = \frac{0.1}{N}$, so that only half of the government's revenue from the estate tax is transferred back to the households. According to equation (4.2), this decreases the absolute value of the parameters $\tilde{\alpha}_k$, thus decreasing cross-sectional mean reversion. In the second alternate parameterization, the standard deviation of households' individual-specific risky asset returns is increased to $\kappa = 0.25$.[26] According to equation (3.15), this decreases the value of $\tilde{\sigma}$, thus decreasing household exposure to idiosyncratic investment risk.

In Figure 2 and Table 2, we compare the equilibrium distributions of wealth for these two alternate parameterizations and the baseline parameterization. The figure and table confirm the basic result from Theorem 2.1 and Proposition 3.2—lower reversion rates increase inequality, and lower exposure to idiosyncratic investment risk reduces inequality. Similarly, in Figures 4-5, we compare intergenerational economic mobility for both alternate parameterizations and the baseline parameterization.[27] The second and third rows of Table 3 also report rank correlations for these alternate parameterizations. The figure and the table both confirm the result from Theorem 3.4, since lower reversion rates lead to less economic mobility.

[26]Because it is only the value of $\tilde{\sigma}$ that affects the equilibrium of the model, this alternate parameterization can be reinterpreted as any combination of the parameters λ, κ, and r that yields the same value of $\tilde{\sigma}$.

[27]As always, these results are obtained by averaging the results of 500 simulations of the model.

According to Figure 5 and Table 3, the parameterization of the model with a lower exposure to idiosyncratic investment risk actually leads to less economic mobility than in the baseline parameterization, despite the fact that this parameterization generates a more even equilibrium distribution of wealth. This result is consistent with equation (3.20) from Theorem 3.4, since, according to equation (4.2), a more even distribution of wealth for this alternate parameterization implies less cross-sectional mean reversion than in the baseline parameterization. According to our numerical results, then, the negative effect on economic mobility of this lower mean reversion is larger than the positive effect of less inequality.

Finally, in Figure 6 and the last row of Table 3, we describe intergenerational mobility and rank correlations for a fourth model parameterization that combines the changes in the previous two alternate parameterizations. More specifically, this fourth alternate parameterization has a common lump sum transfer ratio of $\bar{\nu} = \frac{0.5\tau}{N} = \frac{0.1}{N}$ and a standard deviation of households' individual-specific risky asset returns of $\kappa = 0.25$. This parameterization generates an equilibrium distribution of wealth that is nearly identical to the distribution from the baseline parameterization as in Figure 1. However, because household exposure to idiosyncratic investment risk is lower than in the baseline parameterization, equation (3.17) from Proposition 3.2 implies that cross-sectional mean reversion for this parameterization must also be lower than in the baseline. According to Theorem 3.5, then, economic mobility for this alternate parameterization should be lower than in the baseline parameterization, a prediction that is confirmed by Figure 6 and Table 3.

In Figure 8, we plot the average rank over time of a risk-sharing subgroup of 100 households for the baseline parameterization and for the parameterizations of the model with a lower lump-sum transfer ratio and a lower exposure to idiosyncratic investment risk. As in Figure 7, this figure shows the average outcomes for this subgroup in the absence of changes in these households' decisions about consumption or risky-asset allocations. According to Figure 8, in all three parameterizations of the model, the risk-sharing subgroup of households quickly rises in rank and eventually settles somewhere above the median in the equilibrium distribution of wealth. The rank at which the subgroups settle is highest for the high-inequality distribution of the lower lump-sum transfer ratio parameterization and it is lowest for the low-inequality distribution of the lower exposure to idiosyncratic investment risk parameterization. This result is consistent with our discussion of Theorem 3.6 from Section 3.4. In particular, it confirms that the more unequal the equilibrium distribution of wealth is, the higher the risk-sharing subgroup of households rises in the distribution.

5 Conclusion

In this paper, we have introduced new techniques to solve a model in which finitely lived heterogeneous households face uninsurable idiosyncratic investment risk and make optimal decisions about how to consume, invest, and bequest wealth to their offspring. These techniques yield a closed-form rank-by-rank characterization of the equilibrium distribution of wealth that allows for a careful analysis of the determinants of inequality and mobility as well as the implications of increased market completeness in the form of a risk-sharing subgroup of households.

The extent of economic inequality in this setting is determined entirely by two factors—households' exposure to idiosyncratic investment risk and cross-sectional mean reversion as measured by the reversion rates of household wealth. In the steady-state equilibrium, inequality increases in household exposure to investment risk and decreases in mean reversion. We find that economic mobility is decreasing in inequality and increasing in mean reversion. This last result is broadly consistent with the "Great Gatsby curve" (Krueger, 2012) and the empirical results of Corak (2013) and Chetty et al. (2014). We also show that the presence of a risk-sharing subgroup of households raises the rate of wealth accumulation for all households in the economy, and that this subgroup rises higher in a more unequal equilibrium distribution of wealth.

A Proofs

This appendix presents the proofs of Propositions 3.1 and 3.2, Theorems 3.4, 3.5, and 3.6, and Corollary 3.7.

Proof of Proposition 3.1. Under suitable regularity conditions, Itô's Lemma implies that the Hamilton-Jacobi-Bellman equation for the maximization problem of household i born at time $0 \leq t \leq S$ is given by

$$
\begin{aligned}
0 \quad = \max_{c_i(t),\phi_i(t)} & \left\{ e^{-\rho t} \frac{c_i^{1-\gamma}(t)}{1-\gamma} + J_t(w,t) + \frac{1}{2} J_{ww}(w,t) \phi_i^2(t) \kappa^2 w_i^2(t) \right. \\
& \left. + J_w(w,t)[r w_i(t) + (\lambda - r)\phi_i(t) w_i(t) - c_i(t)] \right\},
\end{aligned}
$$

where $J_w(w,t)$ and $J_t(w,t)$ denote respectively the partial derivatives of the value function with respect to wealth w and time t. The first-order conditions for this maximization problem are therefore

$$
c_i^{-\gamma}(t) = e^{\rho t} J_w(w,t), \tag{A.1}
$$

$$
J_w(w,t)(\lambda - r) w_i(t) = -J_{ww}(w,t) \phi_i(t) \kappa^2 w_i^2(t). \tag{A.2}
$$

The next step is to guess and verify the form of the value function $J(w,t)$. We guess that

$$
J(w,t) = e^{-\rho t} a(t) \frac{w_i^{1-\gamma}(t)}{1-\gamma}, \tag{A.3}
$$

where $a(t) > 0$ is a function of time. In this case, the first-order conditions (A.1) and (A.2) imply that

$$
c_i(t) = a^{-\frac{1}{\gamma}}(t) w_i(t), \tag{A.4}
$$

$$
\phi_i(t) = \frac{\lambda - r}{\gamma \kappa^2}. \tag{A.5}
$$

The last step of this proof involves solving for the function $a(t)$ and confirming that the value function from equation (A.3) does indeed satisfy the Hamilton-Jacobi-Bellman equation from above. This is accomplished by substituting the optimal levels of consumption and investment as given by equations (A.4) and (A.5) into the Hamilton-Jacobi-Bellman equation. This yields

$$
\begin{aligned}
0 = a^{1-\frac{1}{\gamma}}(t) \frac{w_i^{1-\gamma}(t)}{1-\gamma} + a(t) w_i^{1-\gamma}(t) \left[r + \frac{(\lambda - r)^2}{\gamma \kappa^2} - a^{-\frac{1}{\gamma}}(t) \right] - a(t) w_i^{1-\gamma}(t) \frac{(\lambda - r)^2}{2\gamma \kappa^2} \\
- \rho a(t) \frac{w_i^{1-\gamma}(t)}{1-\gamma} + \dot{a}(t) \frac{w_i^{1-\gamma}(t)}{1-\gamma},
\end{aligned}
$$

which, after simplifying, implies that

$$0 = \gamma a^{1-\frac{1}{\gamma}}(t) + a(t)\left[(1-\gamma)r - \rho + (1-\gamma)\frac{(\lambda - r)^2}{2\gamma\kappa^2}\right] + \dot{a}(t). \tag{A.6}$$

Using the definition of η from equation (3.5) above, we can rewrite equation (A.6) as

$$0 = \gamma a^{1-\frac{1}{\gamma}}(t) + \gamma\eta a(t) + \dot{a}(t). \tag{A.7}$$

Using the boundary condition $a(S) = \chi(1-\tau)^{1-\gamma}$, it is possible to solve equation (A.7), yielding

$$a(t) = \left(\frac{e^{\eta(S-t)} - 1}{\eta} + \left(\chi(1-\tau)^{1-\gamma}\right)^{\frac{1}{\gamma}} e^{\eta(S-t)}\right)^{\gamma}.$$

Together with equations (A.4) and (A.5), this confirms that both the value function (A.3) and equations (3.3)-(3.5) from Proposition 3.1 are correct. □

Proof of Proposition 3.2. Banner et al. (2005) show that a model defined by the stable version of the rank wealth processes as in equation (3.16) admits a steady-state distribution if and only if $\alpha_1 + \cdots + \alpha_k < 0$, for all $k = 1, \ldots, N$. Furthermore, if such a steady-state distribution exists, it satisfies equation (3.17).

We also wish to confirm that the parameters $\tilde{\alpha}_k$ and $\tilde{\sigma}$ correspond to the rank-based relative growth rates and idiosyncratic volatilities for the modified wealth processes $\tilde{w}_i$. In order to accomplish this, it is necessary first to show that the growth rate of aggregate wealth $\tilde{w}$, which we denote by $\mu(t)$, satisfies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu(t)\,dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{N}\sum_{k=1}^{N} \mu_{p_t(k)}(t)\,dt, \tag{A.8}$$

where $\mu_{p_t(k)}$ refers to the growth rate process for the k-th ranked modified wealth processes $\tilde{w}_i$ as given by equation (3.12). According to Fernholz (2002), Proposition 2.1.2, for all $i = 1, \ldots, N$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu(t)\,dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_i(t)\,dt.$$

In order to characterize the limit of the time-average of the process μ, then, we only need to characterize the limit of the time-average of the processes μ_i. For all $i = 1, \ldots, N$,

$$\mu_i(t) = \sum_{k=1}^{N} \mu_{p_t(k)}(t) I_{\{0\}}\left(\tilde{\theta}_i(t) - \tilde{\theta}_{(k)}(t)\right),$$

so it follows that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mu_i(t)\,dt$$
$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{k=1}^N \mu_{p_t(k)}(t) I_{\{0\}}\left(\tilde{\theta}_i(t) - \tilde{\theta}_{(k)}(t)\right)\,dt. \tag{A.9}$$

The ex-ante symmetry across households in this setup implies that the N households spend the same fraction of time as each other in each rank. In other words, for all $k = 1, \ldots, N$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T I_{\{0\}}\left(\tilde{\theta}_i(t) - \tilde{\theta}_{(k)}(t)\right)\,dt = \frac{1}{N},$$

which together with equation (A.9) implies that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{N}\sum_{k=1}^N \mu_{p_t(k)}(t)\,dt.$$

According to equation (A.8), the rank-based relative growth rates $\tilde{\alpha}_k$, $k = 1, \ldots, N-1$, are given by

$$\tilde{\alpha}_k = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\psi\left(t - \bar{t}(t)S\right) + \frac{1_Z(t)}{t_0}\log\left(1 - \tau + \frac{\nu_{(k)}\left(\bar{t}(t)S\right)}{\tilde{w}_{(k)}\left(\bar{t}(t)S\right)}\right)\right.$$
$$\left. - \frac{1}{N}\sum_{l=1}^N \psi\left(t - \bar{t}(t)S\right) - \frac{1}{N}\sum_{\ell=1}^N \frac{1_Z(t)}{t_0}\log\left(1 - \tau + \frac{\nu_{(\ell)}\left(\bar{t}(t)S\right)}{\tilde{w}_{(\ell)}\left(\bar{t}(t)S\right)}\right)\right]\,dt,$$

which simplifies to

$$\tilde{\alpha}_k = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1_Z(t)}{t_0}\left[\log\left(1 - \tau + \frac{\nu_{(k)}\left(\bar{t}(t)S\right)}{\tilde{w}_{(k)}\left(\bar{t}(t)S\right)}\right) - \frac{1}{N}\sum_{\ell=1}^N \log\left(1 - \tau + \frac{\nu_{(\ell)}\left(\bar{t}(t)S\right)}{\tilde{w}_{(\ell)}\left(\bar{t}(t)S\right)}\right)\right]\,dt$$
$$= \lim_{T\to\infty}\frac{1}{T}\sum_{j=1}^{\bar{t}(T)}\int_{jS}^{jS+t_0}\frac{1}{t_0}\left[\log\left(1 - \tau + \frac{\nu_{(k)}\left(jS\right)}{\tilde{w}_{(k)}\left(jS\right)}\right) - \frac{1}{N}\sum_{\ell=1}^N \log\left(1 - \tau + \frac{\nu_{(\ell)}\left(jS\right)}{\tilde{w}_{(\ell)}\left(jS\right)}\right)\right]\,dt$$
$$= \lim_{j\to\infty}\frac{1}{S}\log\left(1 - \tau + \frac{\nu_{(k)}\left(jS\right)}{\tilde{w}_{(k)}\left(jS\right)}\right) - \frac{1}{SN}\sum_{\ell=1}^N \log\left(1 - \tau + \frac{\nu_{(\ell)}\left(j\bar{t}\right)}{\tilde{w}_{(\ell)}\left(jS\right)}\right). \tag{A.10}$$

Equation (A.10), which confirms equation (3.14), uses the definition of $\bar{t}(t)$ from equation (3.10) and the fact that $Z = \cup_{j=1}^{\infty}[jS, jS + t_0]$. Similarly, the common idiosyncratic volatility

$\tilde{\sigma}$ is given by

$$\tilde{\sigma}^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T 2 \left(\frac{\lambda - r}{\gamma\kappa} \right)^2 dt = 2 \left(\frac{\lambda - r}{\gamma\kappa} \right)^2,$$

which confirms equation (3.15). □

Proof of Theorem 3.4. Banner et al. (2005) show that, if our setup admits a steady-state equilibrium, then the process $\log\theta^*_{(k)} - \log\theta^*_{(k+1)}$ behaves asymptotically like a Brownian motion that is reflected at the origin and has negative drift $2(\alpha_1 + \cdots + \alpha_k)$. The expected value of the stopping time at zero for a Brownian motion that is reflected at the origin and has negative drift is simply equal to its initial value divided by the drift rate. This implies that $S_k(t)$, the stopping time at which the $k+1$-th wealthiest household in the economy overtakes the k-th wealthiest household, satisfies

$$E\left[S_k(t) \;\middle|\; \theta^*_{(k)}(t), \theta^*_{(k+1)}(t)\right] = \frac{\log\theta^*_{(k)}(t) - \log\theta^*_{(k+1)}(t)}{-2(\alpha_1 + \cdots + \alpha_k)}, \tag{A.11}$$

in the steady-state equilibrium. According to Theorem 2.1, in the steady-state equilibrium it is also the case that

$$E\left[\log\theta^*_{(k)}(t) - \log\theta^*_{(k+1)}(t)\right] = \frac{\sigma^2}{-4(\alpha_1 + \cdots + \alpha_k)}. \tag{A.12}$$

As a consequence, equations (A.11) and (A.12) imply that

$$E[S_k(t)] = \frac{\sigma^2}{8(\alpha_1 + \cdots + \alpha_k)^2},$$

which completes the proof. □

Proof of Theorem 3.5. According to equations (3.16) and (3.22), the processes $w^{**}_{(k)}$ are related to the processes $w^*_{(k)}$ by

$$w^{**}_{(k)}(t) = w^*_{(k)}(xt),$$

for all $k = 1, \ldots, N$ and $t > 0$. As a consequence, the events $\{w^{**}_i(t) = w^{**}_{(k)}(t)\}$ are equivalent to the corresponding events $\{w^{**}_i(xt) = w^{**}_{(k)}(xt)\}$, for all $i, k = 1, \ldots, N$ and $t > 0$. In the steady state, the expected stopping times $T^*_{k,l}(t)$ and $T^{**}_{k,l}(t)$ are time invariant, so it follows that

$$E[T^*_{k,l}(t)] = xE[T^{**}_{k,l}(t)],$$

which confirms equation (3.25). □

Proof of Theorem 3.6. If the households in the risk-sharing subgroup do not change their consumption or risky-asset allocations, then the dynamics of the stable versions of the wealth processes w_i^* are given by

$$d\log w_i^*(t) = \begin{cases} \left[\frac{1}{m}\sum_{j=1}^m \tilde{\alpha}_{r_t(j)} + \left(1-\frac{1}{m}\right)\frac{\tilde{\sigma}^2}{4}\right] dt + \frac{1}{m}\sum_{j=1}^m \frac{\tilde{\sigma}}{\sqrt{2}}\, dB_j(t) & \text{for } i=1,\dots,m \\ \tilde{\alpha}_{r_t(i)}\, dt + \frac{\tilde{\sigma}}{\sqrt{2}}\, dB_i(t) & \text{for } i=m+1,\dots,N \end{cases}. \tag{A.13}$$

Note that the wealth dynamics for those households in the risk-sharing subgroup are derived using Itô's Lemma as in Fernholz (2002), Proposition 1.1.15. Let w^m denote the total wealth holdings of all households in the risk-sharing subgroup, so that

$$w^m(t) = w_1^*(t) + \cdots + w_m^*(t),$$

and let w^c denote the total wealth holdings of all households outside the risk-sharing subgroup, so that

$$w^c(t) = w_{m+1}^*(t) + \cdots + w_N^*(t).$$

If the occupation times $\zeta_1, \dots, \zeta_{N-m+1}$ are all positive in equilibrium, then it follows that

$$\lim_{T\to\infty} \frac{1}{T}\log w^m(T) = \lim_{T\to\infty} \frac{1}{T}\log w^c(T), \tag{A.14}$$

otherwise either $\zeta_1 = 1$ or $\zeta_{N-m+1} = 1$. Using arguments similar to those in the proof of Proposition 3.2, it can be shown that

$$\lim_{T\to\infty} \frac{1}{T}\log w^m(T) = \lim_{T\to\infty} \frac{1}{T}\log w_i^*(T),$$

for $i = 1, \dots, m$, and that

$$\lim_{T\to\infty} \frac{1}{T}\log w^c(T) = \lim_{T\to\infty} \frac{1}{T}\log w_i^*(T),$$

for $i = m+1, \dots, N$. Using equation (A.13), then, it follows that equation (A.14) implies that

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[\sum_{k=1}^{N-m+1}\sum_{j=k}^{k+m-1} 1_{\{r_t(1)=k\}}(t)\frac{\tilde{\alpha}_j}{m} + \left(1-\frac{1}{m}\right)\frac{\tilde{\sigma}^2}{4}\right] dt = \\ \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{k=1}^{N-m+1}\sum_{j<k,\, j>k+m-1} 1_{\{r_t(1)=k\}}(t) 1_{\{r_t(i)=j\}}(t)\tilde{\alpha}_j\, dt, \end{aligned} \tag{A.15}$$

where $m+1 \le i \le N$. By symmetry,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{\{r_t(1)=k\}}(t) 1_{\{r_t(i)=j\}}(t)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{\{r_t(1)=k\}}(t) \frac{1}{N-m},$$

for all $m+1 \le i \le N$, $k = 1, \ldots, N-m+1$, $j < k$, and $j > k+m-1$, and in steady state

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{\{r_t(1)=k\}}(t) dt = E[1_{\{r_t(1)=k\}}(t)] = \zeta_k,$$

for all $k = 1, \ldots, N-m+1$. Furthermore, we know that $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_N = 0$, so it follows that equation (A.15) can be simplified to yield

$$\frac{1}{m} \sum_{k=1}^{N-m+1} \zeta_k \left(\tilde{\alpha}_k + \cdots + \tilde{\alpha}_{k+m-1}\right) + \left(1 - \frac{1}{m}\right) \frac{\tilde{\sigma}^2}{4} = -\frac{1}{N-m} \sum_{k=1}^{N-m+1} \zeta_k \left(\tilde{\alpha}_k + \cdots + \tilde{\alpha}_{k+m-1}\right).$$

After further simplification, this yields

$$\sum_{k=1}^{N-m+1} \zeta_k \left(\tilde{\alpha}_k + \cdots + \tilde{\alpha}_{k+m-1}\right) = \frac{(1-m)(N-m)}{4N} \tilde{\sigma}^2, \tag{A.16}$$

which confirms equation (3.28).

Because $\tilde{\alpha}_{N-m+1} + \cdots + \tilde{\alpha}_N > 0$, it follows that if no solution to equation (A.16) exists, then it must be that the left-hand side of this equation is greater than the right-hand side for all possible values of the occupation times ζ_k. In this case, however, it is also true that the left-hand side of equation (A.15) is greater than the right-hand side for all possible values of ζ_k as well, which implies that $\lim_{T\to\infty} \frac{1}{T} \log w^m(T) > \lim_{T\to\infty} \frac{1}{T} \log w^c(T)$. Of course, this means that the households in the risk-sharing subgroup are diverging from the rest of the households in the economy, and hence $\zeta_1 \to 1$. □

Proof of Corollary 3.7. This proof is essentially the same as the proof of Theorem 3.6, except that now the dynamics of the stable versions of the wealth processes w_i^* are given by

$$d \log w_i^*(t) = \begin{cases} \left[\frac{1}{m} \sum_{j=1}^m \tilde{\alpha}_{r_t(j)} + \left(1 - \frac{1}{m}\right) \frac{m\tilde{\sigma}^2}{4} + \Delta\right] dt + \frac{1}{m} \sum_{j=1}^m \frac{\sqrt{m}}{\sqrt{2}} \tilde{\sigma}\, dB_j(t) & \text{for } i = 1, \ldots, m \\ \tilde{\alpha}_{r_t(i)}\, dt + \frac{\tilde{\sigma}}{\sqrt{2}}\, dB_i(t) & \text{for } i = m+1, \ldots, N \end{cases}. \tag{A.17}$$

There are two differences between equation (A.17) and equation (A.13), both the result of households in the risk-sharing subgroup altering their behavior in response to their new investment options. First, the term Δ enters into the growth rate in equation (A.17). Second, the standard deviation is multiplied by $\sqrt{m}$ in equation (A.17) (and this also multiplies the extra growth rate term that appears via Itô's Lemma).

The standard deviation is multiplied by $\sqrt{m}$ in equation (A.17) because the optimal share of wealth invested in the risky asset for those households in the risk-sharing subgroup is now given by

$$\phi_i'(t) = \frac{m(\lambda - r)}{\gamma\kappa^2}.$$

This implies that the standard deviation of the return on wealth for those households is equal to

$$\frac{\kappa}{\sqrt{m}}\left(\frac{m(\lambda - r)}{\gamma\kappa^2}\right) = \frac{\sqrt{m}}{\sqrt{2}}\tilde{\sigma}.$$

In this case, the share of wealth $\phi'(t)$ is invested in equal shares in each of the m individual-specific risky assets that are available to the households in the risk-sharing subgroup.

Similarly, to see why Δ enters into equation (A.17), note that according to equation (3.8), the growth rate of wealth for households in the risk-sharing subgroup are now given by

$$\psi'(t) = r + \frac{m(2\gamma - 1)(\lambda - r)^2}{2\gamma^2\kappa^2} - \left(\frac{e^{\eta'(S-t)} - 1}{\eta'} + (\chi(1-\tau)^{1-\gamma})^{\frac{1}{\gamma}} e^{\eta'(S-t)}\right)^{-1},$$

for $0 \leq t \leq S$, where

$$\eta' = \frac{(1-\gamma)r - \rho}{\gamma} + \frac{m(1-\gamma)(\lambda - r)^2}{2\gamma^2\kappa^2} = \eta + \frac{(m-1)(1-\gamma)(\lambda - r)^2}{2\gamma^2\kappa^2}.$$

It follows that the modified wealth processes $\tilde{w}_i$, $i = 1, \ldots, m$, in this case follow the stochastic differential equation

$$d\log\tilde{w}_i(t) = \psi'\left(t - \bar{t}(t)S\right)\, dt + \frac{1_Z(t)}{t_0}\log\left(1 - \tau + \frac{\nu_i\left(\bar{t}(t)S\right)}{\tilde{w}_i\left(\bar{t}(t)S\right)}\right)\, dt + \frac{1}{m}\sum_{j=1}^{m}\sqrt{m}\left(\frac{\lambda - r}{\gamma\kappa}\right)\, dB_j(t).$$

By integrating the growth rate processes ψ and ψ', it is not hard to show that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\psi'\left(t - \bar{t}(t)S\right) - \psi\left(t - \bar{t}(t)S\right)\right]\, dt &= \frac{(m-1)(\lambda - r)^2}{2\gamma\kappa^2} + \frac{1}{S}\log\left[\frac{\eta'\left((1+\eta\chi^{\frac{1}{\gamma}})e^{\eta S} - 1\right)}{\eta\left((1+\eta'\chi^{\frac{1}{\gamma}})e^{\eta' S} - 1\right)}\right] \\ &= \Delta,\end{aligned}$$

which yields stable versions of the wealth processes w_i^* that match equation (A.17).

Using similar arguments as in the proof of Theorem 3.6, then, equation (A.17) implies that

$$\frac{1}{m}\sum_{k=1}^{N-m+1}\zeta_k\left(\tilde{\alpha}_k+\cdots+\tilde{\alpha}_{k+m-1}\right)+(m-1)\frac{\tilde{\sigma}^2}{4}+\Delta=-\frac{1}{N-m}\sum_{k=1}^{N-m+1}\zeta_k\left(\tilde{\alpha}_k+\cdots+\tilde{\alpha}_{k+m-1}\right),$$

which after simplification yields

$$\sum_{k=1}^{N-m+1}\zeta_k\left(\tilde{\alpha}_k+\cdots+\tilde{\alpha}_{k+m-1}\right)=\frac{m(1-m)(N-m)}{4N}\tilde{\sigma}^2-\frac{m(N-m)}{N}\Delta,$$

thus confirming equation (3.30). □

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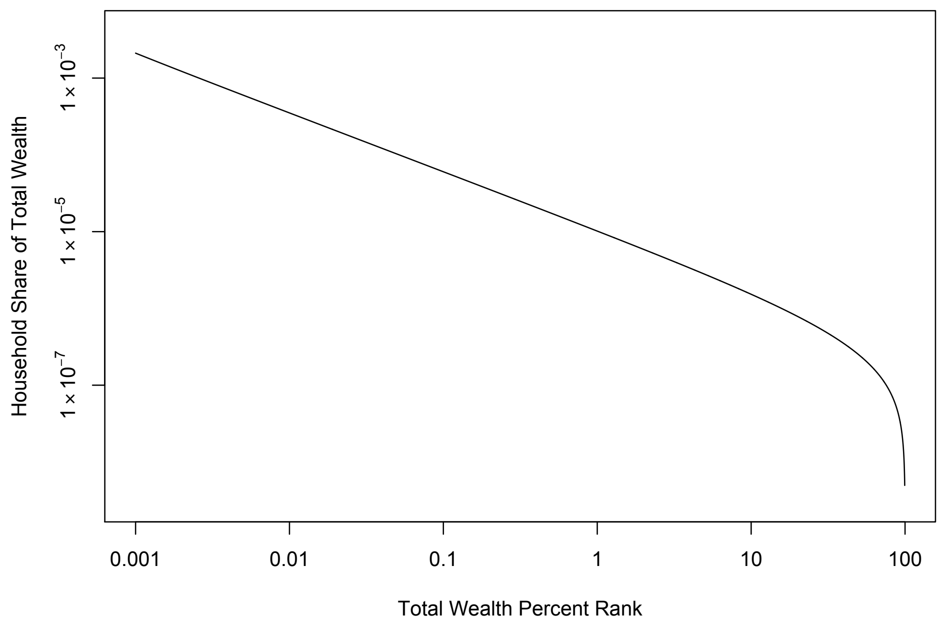

Figure 1: Individual households' wealth shares for the baseline parameterization of the model.

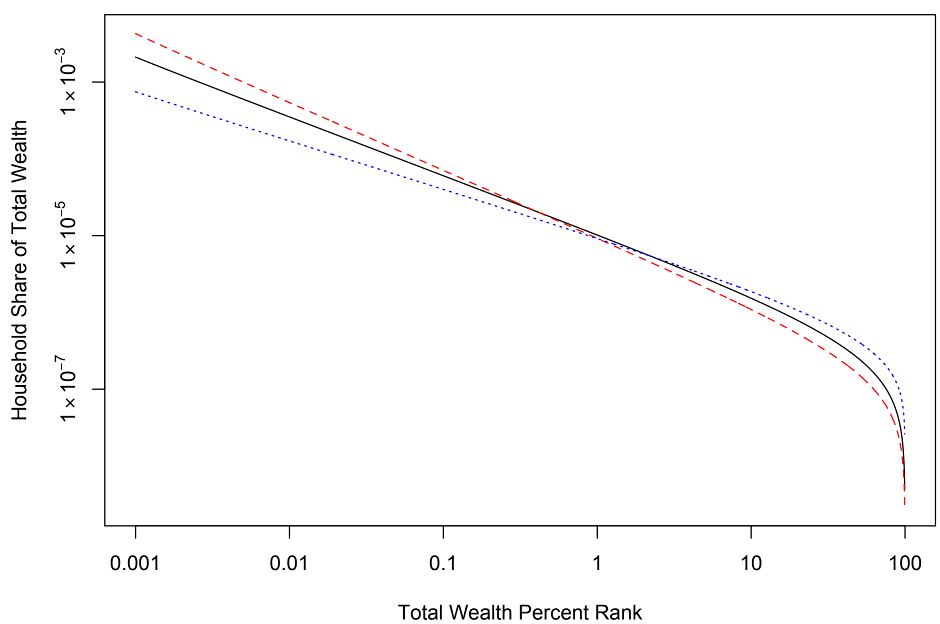

Figure 2: Individual households' wealth shares for three parameterizations of the model: baseline (solid black line), lower lump-sum transfer ratio (dashed red line), and lower exposure to idiosyncratic investment risk (dotted blue line).

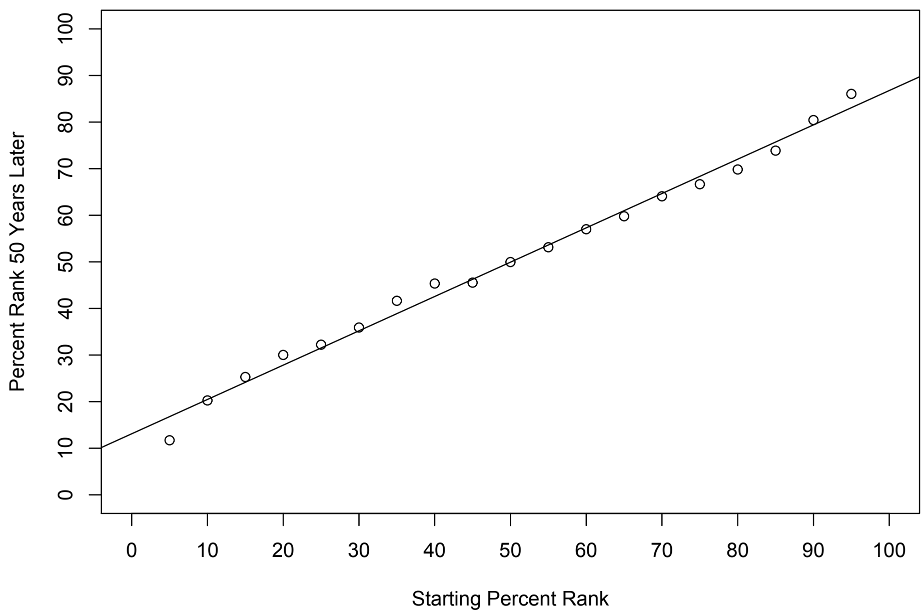

Figure 3: Intergenerational mobility (50 years) for the baseline parameterization of the model. The rank-rank slope and intercept are 0.737 and 13.10, respectively.

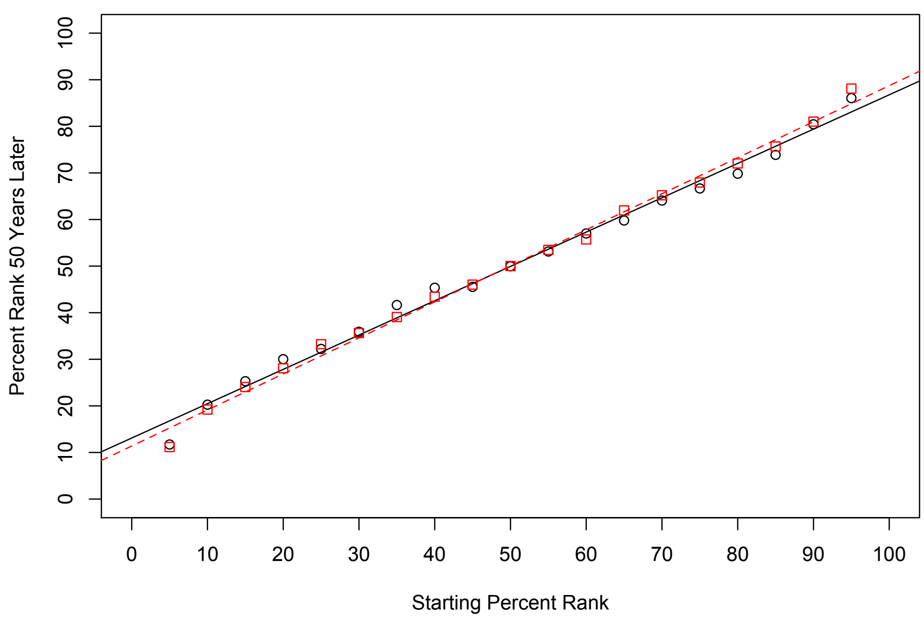

Figure 4: Intergenerational mobility (50 years) for two parameterizations of the model: baseline (solid black line, rank-rank slope of 0.737 and intercept of 13.10) and lower lump-sum transfer ratio (dashed red line, rank-rank slope of 0.774 and intercept of 11.38).

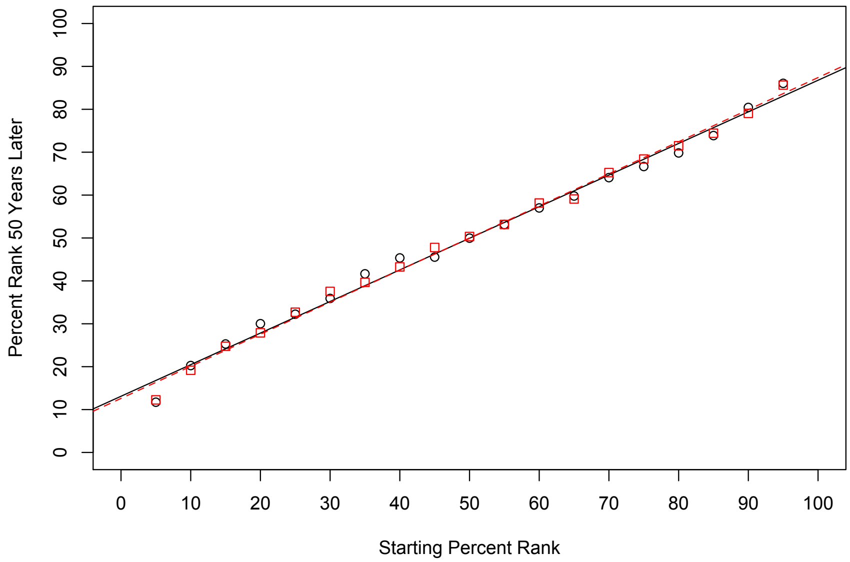

Figure 5: Intergenerational mobility (50 years) for two parameterizations of the model: baseline (solid black line, rank-rank slope of 0.737 and intercept of 13.10) and lower exposure to idiosyncratic investment risk (dashed red line, rank-rank slope of 0.748 and intercept of 12.60).

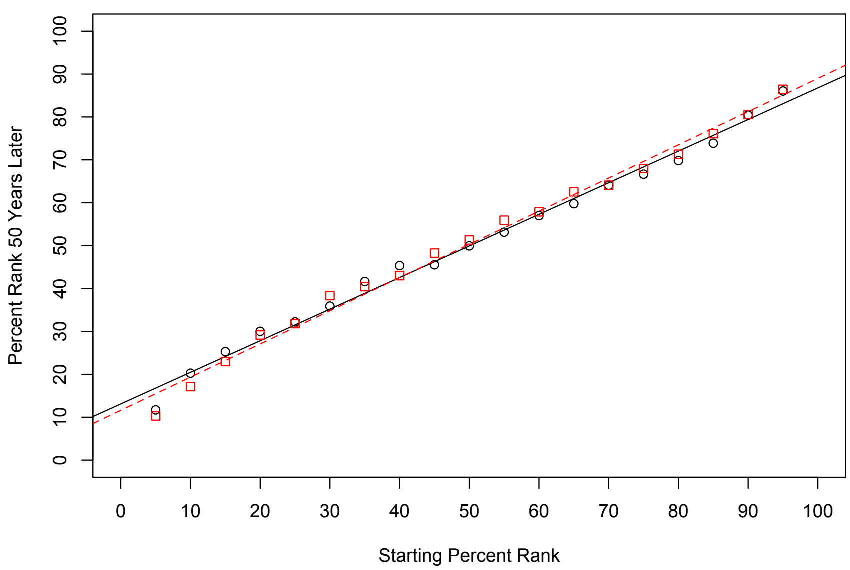

Figure 6: Intergenerational mobility (50 years) for two parameterizations of the model: baseline (solid black line, rank-rank slope of 0.737 and intercept of 13.10) and both lower lump-sum transfer ratio and lower exposure to idiosyncratic investment risk (dashed red line, rank-rank slope of 0.774 and intercept of 11.61).

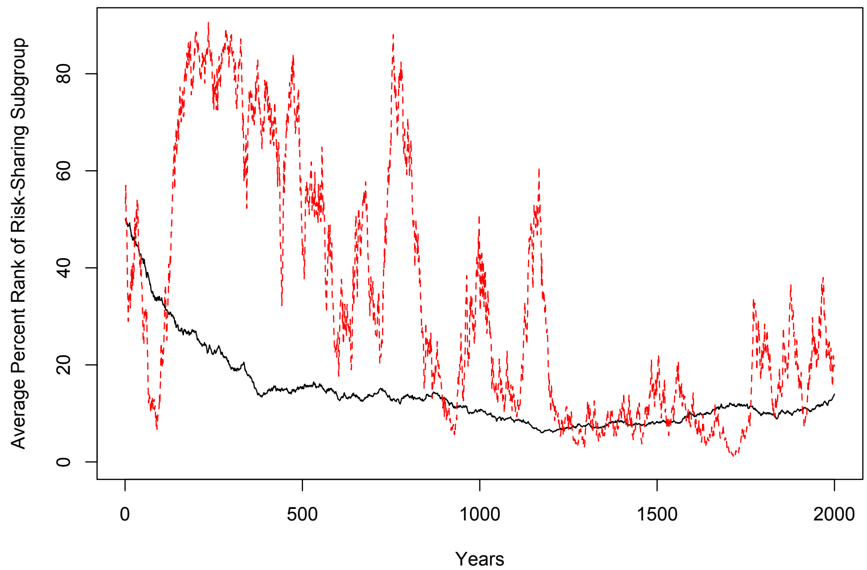

Figure 7: The average percent rank of the households in the risk-sharing subgroup over time (solid black line) and for an individual household with no risk sharing (dashed red line), for the baseline parameterization of the model.

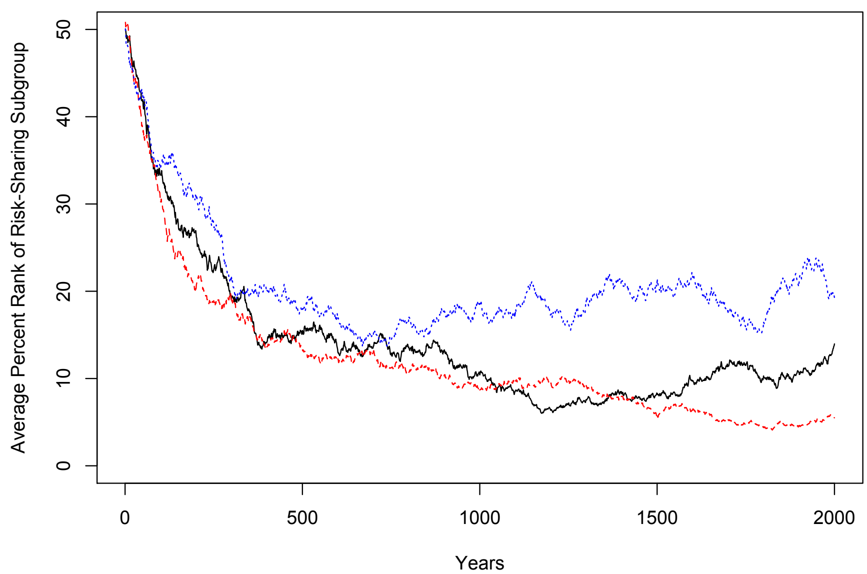

Figure 8: The average percent rank of the households in the risk-sharing subgroup over time for three parameterizations of the model: baseline (solid black line), lower lump-sum transfer ratio (dashed red line), and lower exposure to idiosyncratic investment risk (dotted blue line).

Household Wealth Percent Rank	Model Wealth Shares	U.S. Wealth Shares
0-0.01	11.6%	11.2%
0.01-0.1	10.7%	10.8%
0.1-0.5	11.7%	12.5%
0.5-1	6.5%	7.3%
1-10	29.5%	35.4%
10-100	29.8%	22.8%

Table 1: Household wealth shares for the baseline parameterization of the model and for the 2012 U.S. wealth distribution.

Household Wealth Percent Rank	Baseline Parameterization	Lower Lump-Sum Transfer Ratio	Lower Exposure to Idiosyncratic Investment Risk
0-0.01	11.6%	24.1%	4.2%
0.01-0.1	10.7%	14.3%	6.2%
0.1-0.5	11.7%	12.5%	8.8%
0.5-1	6.5%	6.1%	5.6%
1-10	29.5%	23.5%	31.3%
10-100	29.8%	19.5%	44.2%

Table 2: Household wealth shares for three parameterizations of the model.

Parameterization	10-Year Rank Correlation	20-Year Rank Correlation
Baseline	0.9486	0.9005
Alternate 1	0.9530	0.9089
Alternate 2	0.9489	0.9011
Alternate 3	0.9560	0.9145

Table 3: Rank correlations for four parameterizations of the model: baseline, lower lump-sum transfer ratio (Alternate 1), lower exposure to idiosyncratic investment risk (Alternate 2), and both lower lump-sum transfer ratio and lower exposure to idiosyncratic investment risk (Alternate 3).